Formal punctured ribbons and two-dimensional local fields

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Abstract

We investigate formal ribbons on curves. Roughly speaking, formal ribbon is a family of locally linearly compact vector spaces on a curve. We establish a one-to-one correspondence between formal ribbons on curves plus some geometric data and some subspaces of two-dimensional local field.

1 Introduction

The aim of this paper is to obtain an appropriate generalization of the Krichever map for algebraic surfaces.

We recall that in the classical 1-dimensional case it is a one-to-one correspondence between integer projective curves over a field k plus line bundles (or torsion free sheaves if the curve is singular) plus some additional data (a distinguished point p of the curve plus a formal local parameter at p, and a formal trivialization at p of the sheaf) and Schur pairs, i.e. pairs of k-subspaces (W, A) of the vector space $V = k((z))$ satisfying a Fredholm condition with respect to the subspace $V_+ = k[[z]]$ (i.e. the complex $W \to V/V_+$ as well as the complex $A \to V/V_+$ has to be Fredholm) such that A is a k-subalgebra of V and $A \cdot W \subset W$.

Parshin and Osipov established the Krichever correspondence in higher dimensions (see [15], [12], also [14], [17]). In the 2-dimensional case it starts with a "flag" $(X \supset C \supset p)$ (where X is a projective algebraic surface over a field k, C is an ample curve, p is a k-point, X and C are smooth at p), a vector bundle $\mathcal{F}$ of rank r on X plus formal trivialization e_p of $\mathcal{F}$ at p, and formal local parameters u, t at p. By these data this correspondence associates the k-subalgebra A of $V = k((u))((t))$ and k-subspace W of $V^{\oplus r}$ with Fredholm condition for all i for $(A \cap t^i V)/(A \cap t^{i+1} V)$ as a subspace of $k((u))$ with respect to $k[[u]]$, and with Fredholm condition for all i for $(W \cap t^i V^{\oplus r})/(W \cap t^{i+1} V^{\oplus r})$ as a subspace of $k((u))^{\oplus r}$ with respect to $k[[u]]^{\oplus r}$. A is the

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image of the structure sheaf $\mathcal{O}_X$, and W is the image of the sheaf $\mathcal{F}$. If X is Cohen-Macaulay surface, C is an ample Cartier divisor on X, then the pair (A, W) contains all information about $(X, C, p, \mathcal{F}, e_p, u, t)$, see [15, theorem 4] and [12, theorem 6].

However there was a problem, because contrary to the 1-dimensional case it is not true that any such pair of subspaces comes from geometric data. To solve this problem we introduce another type of geometric objects which we call "ribbons" (or more exactly "formal ribbons", but we will omit in the sequel the word "formal"). This terminology comes from [7], where a similar object was defined[1]. We decompose the Krichever map into the composition of the following maps

$$\left\{ \begin{array}{c} \text{geometric data} \\ (X, C, p, \mathcal{F}, e_p, u, t) \end{array} \right\} \subset \left\{ \begin{array}{c} \text{geometric data} \\ \text{on ribbons} \end{array} \right\} \mapsto \left\{ \begin{array}{c} \text{pairs of subspaces } (W, A) \\ \text{with Fredholm conditions} \end{array} \right\}$$

Ribbons are ringed spaces which are, on the one hand side, more general as the notion of "formal scheme" of Grothendieck, on the other hand side, they have some extra features. We explain them exactly in section 2.

In section 3 we clarify the cohomology of sheaves, which we call ind-pro-quasicoherent sheaves on a ribbon. We investigate the coherence property of ribbons.

In section 4 we clarify the structure of the Picard group of a ribbon.

In section 5 we establish a one-to-one correspondence between the classes of isomorphic "geometric data" (punctured ribbon plus torsion free sheaf on it plus some extra data) and the "Schur pairs" $(A, W) \subset (V, V^{\oplus r})$, were A is a k-subalgebra of V, and $A \cdot W \subset W$, satisfying Fredholm conditions for the subquotients (as explained above).

We computed also several examples to illustrate the general theory.

We note that families of Tate spaces (i.e. of locally linearly compact vector spaces) were studied also in [5].

We think that the ribbons and geometric data on them, which are introduced in this paper, will help to find geometric solutions of generalizations of Parshin's two-dimensional KP-hierarchy, see [14], [18].

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2 The category of ribbons

2.1 Definition of a ribbon.

Let S be a Noetherian base scheme.

Definition 1. A ribbon $(C, \mathcal{A})$ over S is given by the following data.

[1] More precisely, our ribbons are more general: the ribbons from [7] are $(C, \mathcal{A}_0)$ in our terminology

1. A flat family of reduced algebraic curves $\tau : C \to S$.

2. A sheaf $\mathcal{A}$ of commutative $\tau^{-1}\mathcal{O}_S$-algebras on C.

3. A descending sheaf filtration $(\mathcal{A}_i)_{i\in\mathbb{Z}}$ of $\mathcal{A}$ by $\tau^{-1}\mathcal{O}_S$-submodules which satisfies the following axioms:

   (a) $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$, $1 \in \mathcal{A}_0$ (thus $\mathcal{A}_0$ is a subring, and for any $i \in \mathbb{Z}$ the sheaf $\mathcal{A}_i$ is a $\mathcal{A}_0$-submodule);

   (b) $\mathcal{A}_0/\mathcal{A}_1$ is the structure sheaf $\mathcal{O}_C$ of C;

   (c) for each i the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}$ (which is a $\mathcal{A}_0/\mathcal{A}_1$-module by (3a)) is a coherent sheaf on C, flat over S, and for any $s \in S$ the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}\mid_{C_s}$ has no coherent subsheaf with finite support, and is isomorphic to $\mathcal{O}_{C_S}$ on a dense open set;

   (d) $\mathcal{A} = \varinjlim\limits_{i\in\mathbb{Z}} \mathcal{A}_i$, and $\mathcal{A}_i = \varprojlim\limits_{j>0} \mathcal{A}_i/\mathcal{A}_{i+j}$ for each i.

Remark 1. It follows from (3c) of the definition that if C_s (for $s \in S$) is an irreducible curve, then the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}\mid_{C_s}$ is a torsion free sheaf on C_s for any $i \in \mathbb{Z}$.

Notation 1. For simplicity we denote a ribbon $(C,\mathcal{A})$ over $\operatorname{Spec} R$, where R is a ring, as a ribbon over R.

Example 1. If X is an algebraic surface over a field k, and $C \subset X$ is a reduced effective Cartier divisor, we obtain a ribbon $(C,\mathcal{A})$ over k, where

$$\mathcal{A} := \mathcal{O}_{\hat{X}_C}(*C) = \varinjlim_{i\in\mathbb{Z}} \mathcal{O}_{\hat{X}_C}(-iC) = \varinjlim_{i\in\mathbb{Z}} \varprojlim_{j\geq 0} J^i/J^{i+j}$$

$$\mathcal{A}_i := \mathcal{O}_{\hat{X}_C}(-iC) = \varprojlim_{j\geq 0} J^i/J^{i+j}, \quad i \in \mathbb{Z},$$

where $\hat{X}_C$ is the formal scheme which is the completion of X at C, and J is the ideal sheaf of C on X (the sheaf J is an invertible sheaf).

Proposition 1. *1. For any $i \geq 0$ the ringed space $X_i = (C, \mathcal{A}_0/\mathcal{A}_{i+1})$, $(i \geq 0)$ is a scheme, which is flat over S.*

2. For any $j \in \mathbb{Z}$ and any $i \geq 0$ the sheaf $\mathcal{A}_j/\mathcal{A}_{j+i+1}$ is a coherent sheaf on X_i, which is a flat sheaf over S.

3. If $X_\infty = (C,\mathcal{A}_0)$, then X_∞ is a locally ringed space, and we have closed embeddings of schemes

$$X_0 \subset X_1 \subset X_2 \subset \dots X_i \subset X_{i+1} \subset \dots$$

such that $X_\infty = \varinjlim\limits_{i\geq 0} X_i$ in the category of locally ringed spaces.

Proof. We prove the first statement of the proposition.

At first, we show that X_i are locally ringed spaces. By definition, we have that X_0 is the scheme $(C, \mathcal{O}_C)$. Therefore X_0 is a locally ringed space. We have that for every $i \geq 0$ the subsheaf $\mathcal{A}_i/\mathcal{A}_{i+1} \subset \mathcal{A}_0/\mathcal{A}_{i+1}$ is a nilpotent ideal sheaf because of (3a) of definition 1. We consider the following exact triple of sheaves on C:

$$0 \longrightarrow \mathcal{A}_i/\mathcal{A}_{i+1} \longrightarrow \mathcal{A}_0/\mathcal{A}_{i+1} \xrightarrow{\pi_i} \mathcal{A}_0/\mathcal{A}_i \longrightarrow 0. \tag{1}$$

For each point $P \in C$ we consider the stalks at P of each sheaf from this sequence. We obtain the following exact sequence:

$$0 \longrightarrow (\mathcal{A}_i/\mathcal{A}_{i+1})_P \longrightarrow (\mathcal{A}_0/\mathcal{A}_{i+1})_P \xrightarrow{(\pi_i)_P} (\mathcal{A}_0/\mathcal{A}_i)_P \longrightarrow 0. \tag{2}$$

We apply now induction arguments on i. By induction hypothesis, we assume that the ring $(\mathcal{A}_0/\mathcal{A}_i)_P$ is a local ring with the maximal ideal $\mathcal{M}$. Let $\mathcal{M}'$ be the ideal $\pi_i^{-1}(\mathcal{M})$. Then this ideal is a unique maximal ideal in $(\mathcal{A}_0/\mathcal{A}_{i+1})_P$. Therefore this ring is a local ring. Indeed, if $a \in (\mathcal{A}_0/\mathcal{A}_{i+1})_P \backslash \mathcal{M}'$, then a must be invertible in the ring $(\mathcal{A}_0/\mathcal{A}_{i+1})_P$, since $(\pi_i)_P(a)$ is invertible in the ring $(\mathcal{A}_0/\mathcal{A}_i)_P$, and $(\mathcal{A}_i/\mathcal{A}_{i+1})_P$ is a nilpotent ideal in the ring $(\mathcal{A}_0/\mathcal{A}_{i+1})_P$.

Second, we show that there are natural morphisms $X_i \xrightarrow{\tau_i} S$ of locally ringed spaces for each $i \geq 0$, and that these morphisms are flat. We apply induction on $i \geq 0$. For every $i \geq 0$ the morphism τ_i consists of the topological morphism $\tau : C \to S$ and of morphism of sheaves

$$\tau_i^\sharp \ : \ \mathcal{O}_S \to \tau_*(\mathcal{A}_0/\mathcal{A}_{i+1}), \qquad \text{where}$$
$$\tau_i^\sharp(U) \ : \ \mathcal{O}_S(U) \ni a \longmapsto a \cdot 1 \in \mathcal{A}_0/\mathcal{A}_{i+1}(\tau^{-1}(U))$$

for each open subset $U \subset S$. For each $P \in C$ the morphism

$$(\tau_i^\sharp)_P \ : \ (\mathcal{O}_S)_{\tau(P)} \longrightarrow (\mathcal{A}_0/\mathcal{A}_{i+1})_P$$

is a local morphism, because its composition with the morphism $(\pi_i)_P$ is a local morphism by induction hypothesis.

Now for every $i \geq 0$ the morphism τ_i is a flat morphism by standard results on flat modules (see e.g. [10, ch. 2, §3]), because for each $P \in C$ the $(\mathcal{O}_S)_{\tau(P)}$-modules $(\mathcal{A}_i/\mathcal{A}_{i+1})_P$ and $(\mathcal{A}_0/\mathcal{A}_i)_P$ are flat $(\mathcal{O}_S)_{\tau(P)}$-modules by induction hypothesis on i and by (3c) of definition 1. Therefore we obtain from exact sequence (2) that $(\mathcal{A}_0/\mathcal{A}_{i+1})_P$ is a flat $(\mathcal{O}_S)_{\tau(P)}$-module.

At third, we show that a locally ringed space X_i is scheme for each $i \geq 0$. We consider any affine open subset $U \subset C$. The sequence (1) leads to the following exact triple:

$$0 \to \mathcal{A}_i/\mathcal{A}_{i+1}(U) \longrightarrow \mathcal{A}_0/\mathcal{A}_{i+1}(U) \xrightarrow{\pi} \mathcal{A}_0/\mathcal{A}_i(U) \longrightarrow 0. \tag{3}$$

This sequence is an exact sequence, because the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}$ is a coherent sheaf on C, and U is an affine set. We have that $\mathcal{A}_0/\mathcal{A}_{i+1}(U)$ and $\mathcal{A}_0/\mathcal{A}_i(U)$ are rings, and we are going to show that $(U, (\mathcal{A}_0/\mathcal{A}_{i+1})|_U) \simeq \mathrm{Spec}(\mathcal{A}_0/\mathcal{A}_{i+1}(U))$.

It is clear that the topological space $\underline{\mathrm{Spec}}(\mathcal{A}_0/\mathcal{A}_{i+1}(U)) = U$. Using that $\mathcal{A}_i/\mathcal{A}_{i+1}(U)$ is a nilpotent ideal in the ring $\mathcal{A}_0/\mathcal{A}_{i+1}(U)$, from exact sequence (3), by induction on i we

obtain that the identical map on the ring $\mathcal{A}_0/\mathcal{A}_{i+1}(U)$ induces a well-defined morphism of sheaves on U :

$$\gamma \, : \, \widetilde{\mathcal{A}_0/\mathcal{A}_{i+1}}(U) \longrightarrow (\mathcal{A}_0/\mathcal{A}_{i+1}) \mid_U ,$$

where for any $\mathcal{A}_0/\mathcal{A}_{i+1}(U)$-module N by $\widetilde{N}$ we denote the corresponding quasicoherent sheaf on $\mathrm{Spec}(\mathcal{A}_0/\mathcal{A}_{i+1}(U))$. The map γ is an isomorphism, since it follows from the following exact diagram of sheaves on U:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \widetilde{\mathcal{A}_i/\mathcal{A}_{i+1}}(U) & \longrightarrow & \widetilde{\mathcal{A}_0/\mathcal{A}_{i+1}}(U) & \rightarrow & \widetilde{\mathcal{A}_0/\mathcal{A}_i}(U) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & (\mathcal{A}_i/\mathcal{A}_{i+1})|_U & \longrightarrow & (\mathcal{A}_0/\mathcal{A}_{i+1})|_U & \rightarrow & (\mathcal{A}_0/\mathcal{A}_{i+1})|_U & \longrightarrow 0
\end{array}$$

The left vertical arrow in this diagram is an isomorphism by (3c) of definition 1. The right vertical arrow is an isomorphism by induction on i. Therefore, the middle vertical arrow γ is also an isomorphism. Thus we proved that X_i is a scheme for each $i \ge 0$. It finishes the proof of the first statement of the proposition.

We prove now the second statement of the proposition. As above, the proof is by induction on i. We have the following exact triple of $\mathcal{O}_{X_i}$-modules:

$$0 \longrightarrow \mathcal{A}_{j+i}/\mathcal{A}_{j+i+1} \longrightarrow \mathcal{A}_j/\mathcal{A}_{j+i+1} \longrightarrow \mathcal{A}_j/\mathcal{A}_{j+i} \longrightarrow 0.$$

By definition, the sheaf $\mathcal{A}_{j+i}/\mathcal{A}_{j+i+1}$ is a coherent sheaf on X_i, and a flat sheaf over S. The sheaf $\mathcal{A}_j/\mathcal{A}_{j+i}$ is a coherent $\mathcal{O}_{X_{i-1}}$-module sheaf, and a flat sheaf over S by the induction hypothesis. Therefore, this sheaf is also a coherent $\mathcal{O}_{X_i}$-module sheaf, because both module structures coincide on this sheaf. Thus, the sheaf $\mathcal{A}_j/\mathcal{A}_{j+i+1}$ is a coherent sheaf by [6, prop. 5.7] and flat over S by [6, prop. 9.1]. We proved the second statement of the proposition.

The third statement of the proposition easily follows from exact sequence (1).

□

Definition 2. 1. A morphism φ of ribbons over S

$$\varphi : (C, \mathcal{A}) \to (C', \mathcal{A}')$$

is a morphism of ringed spaces over S that preserves the filtrations, i.e. we have for the map $\varphi^\sharp : \mathcal{A}' \to \varphi_*(\mathcal{A})$, for any $i \in \mathbb{Z}$

$$\varphi^\sharp(\mathcal{A}'_i) \subset \varphi_*(\mathcal{A}_i).$$

2. An isomorphism of ribbons is a morphism that has right and left inverse.

2.2 Base change.

Notation 2. We will also denote the ribbon $(C, \mathcal{A})$ by $\overset{\circ}{X}_\infty$.

For a ribbon $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ over S, and a morphism $\alpha : S' \longrightarrow S$ of Noetherian schemes we define a base change ribbon $\overset{\circ}{X}{}'_\infty = (C', \mathcal{A}')$ over S' in the following way:

$$C' := C \times_S S',$$

$$\mathcal{A}'_j := \varprojlim_{i \geq 1} (\mathcal{A}_j/\mathcal{A}_{j+i}) \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'}$$

for any $j \in \mathbb{Z}$. From statement 2 of proposition 1 we have for any $j \in \mathbb{Z}$, any $i \geq 0$

$$(\mathcal{A}_{j+1}/\mathcal{A}_{j+i+1}) \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'} \subseteq (\mathcal{A}_j/\mathcal{A}_{j+i+1}) \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'}.$$

Therefore we have

$$\ldots \subset \mathcal{A}'_{j+1} \subset \mathcal{A}'_j \subset \mathcal{A}'_{j-1} \subset \ldots$$

and we define

$$\mathcal{A}' := \varinjlim_{i \in \mathbb{Z}} \mathcal{A}'_i.$$

By definition we have $\mathcal{A}'_j/\mathcal{A}'_{j+1} = (\mathcal{A}_j/\mathcal{A}_{j+1}) \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ for any $j \in \mathbb{Z}$, and all axioms from definition of ribbon are satisfied.

Proposition 2. *For the base change $\alpha : S' \longrightarrow S$ we have that for the base change ribbon $\overset{\circ}{X}{}'_\infty = (C', \mathcal{A}')$ the following properties are satisfied.*

1. $X'_i = X_i \times_S S'$ *for any* $i \geq 0$

2. $\mathcal{A}'_j/\mathcal{A}'_{j+i+1} = (\mathcal{A}_j/\mathcal{A}_{j+i+1}) \boxtimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ *for any* $j \in \mathbb{Z}$ *and any* $i \geq 0$

Proof. The proof is clear from definition of a ribbon and proposition 1.

□

3 Coherent sheaves on a ribbon.

3.1 Ind-pro-quasicoherent sheaves.

Definition 3. Let $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ be a ribbon, and $\mathcal{F}$ a sheaf of $\mathcal{A}$-modules. We will call $\mathcal{F}$ *ind-pro-coherent (ind-pro-quasicoherent)* on $\overset{\circ}{X}_\infty$ if it has a descending sheaf filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}}$ with the following properties.

1. $\mathcal{A}_i \mathcal{F}_j \subseteq \mathcal{F}_{i+j}$.

2. $\mathcal{F}_j/\mathcal{F}_{j+1}$ is a coherent (quasicoherent) $\mathcal{O}_C$-module for all j.

3. $\mathcal{F}_i = \varprojlim_j \mathcal{F}_i/\mathcal{F}_{i+j}$.

4. $\mathcal{F} = \varinjlim_i \mathcal{F}_i$.

We recall that projective system $(D_i, i \in \mathbb{N})$ of abelian groups with transition maps $\phi_{i'i}$ satisfies the *ML-condition* (the Mittag-Lefler condition), iff for every $i \in \mathbb{N}$ the descending family of subgroups $\{\phi_{i'i}(D_{i'}) \subset D_i \mid i' \geq i \in \mathbb{N}\}$ will stabilize.

We will need the following lemma, which is easy to prove, using [6, prop. 9.1].

Lemma 1. *If*

$$0 \longrightarrow (K_i) \longrightarrow (A_i) \longrightarrow (B_i) \longrightarrow (C_i) \longrightarrow 0$$

is an exact sequence of projective systems of abelian groups with respect to $\mathbb{N}$, *and projective systems* $(K_i, i \in \mathbb{N})$ *and* $(A_i, i \in \mathbb{N})$ *satisfy the ML-condition, then the induced sequence of projective limits*

$$0 \longrightarrow \varprojlim_{i \in \mathbb{N}} K_i \longrightarrow \varprojlim_{i \in \mathbb{N}} A_i \longrightarrow \varprojlim_{i \in \mathbb{N}} B_i \longrightarrow \varprojlim_{i \in \mathbb{N}} C_i \longrightarrow 0$$

is also exact.

Proposition 3. *Let* $\overset{\circ}{\mathrm{X}}_\infty = (C, \mathcal{A})$ *be a ribbon and* $\mathcal{F}$ *an ind-pro-quasicoherent sheaf on* $\overset{\circ}{\mathrm{X}}_\infty$. *Then we have the following.*

1. $\mathcal{F}_i/\mathcal{F}_{i+j+1}$ *is a quasicoherent* $\mathcal{O}_{X_j}$ *-module for any* $j \geq 0$, $i \in \mathbb{Z}$.
2. *We have that* $\mathcal{F}_i(U)/\mathcal{F}_j(U) \to (\mathcal{F}_i/\mathcal{F}_j)(U)$ *is an isomorphism for all* $i < j$ *and for any affine open subset* $U \subset C$.
3. *If* $\overset{\circ}{\mathrm{X}}_\infty$ *is a ribbon over an Artinian ring, then for any affine open subset* $U \subset C$ *we have* $H^1(U, \mathcal{F}_i) = H^1(U, \mathcal{F}) = 0$.

Proof. The proof of statement 1 of the proposition is analogous to the proof of statement 2 of proposition 1.

We prove statement 2 of the proposition. We always have an exact sequence

$$0 \to \mathcal{F}_j(U) \to \mathcal{F}_i(U) \to (\mathcal{F}_i/\mathcal{F}_j)(U),$$

and we have exact sequences for $i < j < k$

$$0 \to (\mathcal{F}_j/\mathcal{F}_k)(U) \to (\mathcal{F}_i/\mathcal{F}_k)(U) \to (\mathcal{F}_i/\mathcal{F}_j)(U) \to 0,$$

since, by statement 1, $\mathcal{F}_j/\mathcal{F}_k$ is a quasicoherent sheaf of $\mathcal{O}_{X_{k-j-1}}$-modules.

Now since $\mathcal{F}_i(U) = \varprojlim\limits_{k \geq i} (\mathcal{F}_i/\mathcal{F}_k)(U)$ and all maps $(\mathcal{F}_j/\mathcal{F}_{k+1})(U) \to (\mathcal{F}_j/\mathcal{F}_k)(U)$ are surjective, we also have surjections $\mathcal{F}_i(U) \to (\mathcal{F}_i/\mathcal{F}_j)(U)$ (see lemma 1).

We prove statement 3 of the proposition. Since C is a curve over an Artinian ring, every open subset of an affine open set U is again affine. We take an embedding $\mathcal{F}_i \hookrightarrow W$ into a flabby sheaf, then $H^1(U, \mathcal{F}_i)$ is the cokernel of $W(U) \to (W/\mathcal{F})(U)$, and we have to show that any section of $(W/\mathcal{F})(U)$ lifts to a section of $W(U)$.

Since the underlying space U is Noetherian, we have a largest open set $U' \subseteq U$ where a lifting w' of the given section exists. We will show that the assumption $U' \subsetneqq U$ leads to a contradiction. Assume $p \in U \backslash U'$, then we find a neighbourhood $U'' \subset U$ of p

and a lifting w'' on U'' of the given section. If $U' \cap U'' = \varnothing$ we could extend (U', w') to $(U' \cup U'', w'$ on U', w'' on $U'')$. If $U' \cap U'' \neq \varnothing$, we get a section $a = w' - w''$ of $\mathcal{F}_i(U' \cap U'')$.

We *claim* that $\mathcal{F}_i(U') \oplus \mathcal{F}_i(U'') \to \mathcal{F}_i(U' \cap U'')$ is surjective, so we can write $a = a' - a''$ with $a' \in \mathcal{F}_i(U')$, $a'' \in \mathcal{F}_i(U'')$. Then $w|_{U'} = w' - a'$ and $w|_{U''} = w'' - a''$ would give a lifting to $U' \cup U''$, hence U' was non maximal.

Proof of the claim. We have an exact sequence of projective systems

$$\begin{array}{ccccccccc} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & \mathcal{F}_i/\mathcal{F}_{j+1}(U' \cup U'') & \to & \mathcal{F}_i/\mathcal{F}_{j+1}(U') \oplus \mathcal{F}_i/\mathcal{F}_{j+1}(U'') & \to & \mathcal{F}_i/\mathcal{F}_{j+1}(U' \cap U'') & & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 \to & & (\mathcal{F}_i/\mathcal{F}_j)(U' \cup U'') & \to & (\mathcal{F}_i/\mathcal{F}_j)(U') \oplus (\mathcal{F}_i/\mathcal{F}_j)(U'') & \to & (\mathcal{F}_i/\mathcal{F}_j)(U' \cap U'') & & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \end{array}$$

where all transition maps are surjective. Thus the projective limit stays exact (see lemma 1). For $\mathcal{F}$ the assertion follows since cohomology commute with $\varinjlim$.

□

Corollary 1. *Let $\overset{\circ}{\mathrm{X}}_\infty = (C, \mathcal{A})$ be a ribbon over A, where A is an Artinian ring. Let $\mathcal{F}$ be an ind-pro-quasicoherent sheaf on $\overset{\circ}{\mathrm{X}}_\infty$, and C be a projective curve over A.*

1. *If $C = U_1 \cup U_2$, where U_1 and U_2 are affine open subsets, then we have an exact sequence*

$$0 \to H^0(C, \mathcal{F}) \to H^0(U_1, \mathcal{F}) \oplus H^0(U_2, \mathcal{F}) \to H^0(U_1 \cap U_2, \mathcal{F}) \to H^1(C, \mathcal{F}) \to 0.$$

2. *If $\mathcal{F}$ is an ind-pro-coherent sheaf, then*

$$H^*(C, \mathcal{F}) = \varinjlim_{i} \varprojlim_{j \geq i} H^*(X_{j-i}, \mathcal{F}_i/\mathcal{F}_{j+1}).$$

Proof. The first assertion of this corollary is the Mayer-Vietoris exact sequence, due to assertion 3 of proposition 3, because U_1 and U_2 are affine sets.

We prove now the second assertion of this corollary. We note that for any $j \in \mathbb{Z}$ a projective system ($H^0(C, \mathcal{F}_j/\mathcal{F}_{j+i}), i \in \mathbb{N}$) satisfies the ML-condition, because $H^0(C, \mathcal{F}_j/\mathcal{F}_{j+i})$ is an Artinian A-module for any i, j, and the maps in projective system are the maps of A-modules.

We note that, since C is a curve over an Artinian ring, there are some affine open subsets U_1 and U_2 of C such that $C = U_1 \cup U_2$. For any fixed $j \in \mathbb{Z}$ a projective system $(H^0(U_1, \mathcal{F}_j/\mathcal{F}_{j+i}) \oplus H^0(U_2, \mathcal{F}_j/\mathcal{F}_{j+i}), i \in \mathbb{N})$ satisfies the ML-condition because of assertion 2 of proposition 3.

Now, since the cohomology commutes with direct limits, the second assertion of this corollary follows from the first one, using lemma 1.

□

3.2 Coherence property

Remark 2. The sheaf $\mathcal{A}$ may be not coherent in the usual sense (due to H. Cartan, see [16]).

We recall that a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules on a topological space X is coherent if it satisfies the following two properties.

1. $\mathcal{F}$ is locally of finite type, i.e. for any point $x \in X$ there exist an open $U \ni x$ and finite number of sections $s_1, \dots, s_p \in \mathcal{F}(U)$ such that for any $y \in U$ the stalk $\mathcal{F}_y$ is generated by the images of $s_1, \dots, s_p$ over $\mathcal{A}_y$.

2. The sheaf $\mathcal{K} = ker((\mathcal{A}|_U)^{\oplus q} \xrightarrow{(f_1,\dots,f_q)} (\mathcal{F}|_U))$, where $f_i \in \mathcal{F}(U)$ for an open U, is locally of finite type. Here the map $\xrightarrow{(f_1,\dots,f_q)}$ maps an element $(a_1, \dots, a_q)$ to $\sum a_i f_i$.

The sheaf $\mathcal{A}$ is called coherent if it is coherent as $\mathcal{A}$-module.

Let's consider the following ringed space: $(C, \mathcal{O}_C((t))^Q)$, where C is a reduced algebraic curve over a field k, $Q \in C$ is a closed point, and the sheaf $\mathcal{O}_C((t))^Q$ is defined by

$$\mathcal{O}_C((t))^Q(U) := \{\sum_{i=l}^{\infty} c_i t^i, \text{ where } c_i \in \mathcal{O}_C(U) \text{ for } i \geq 0 \quad \text{and } c_i \in \mathcal{J}_Q(U) \text{ for } i < 0 \ \},$$

where $\mathcal{J}_Q$ is the ideal sheaf of the point Q. Clearly, this is a sheaf, and $(C, \mathcal{O}_C((t))^Q)$ is a ribbon over the field k. This sheaf is an analogue of the sheaf $\mathcal{O}_X((t))^{\surd}$ from [8].

Example 2. This is an example of non-coherent sheaf $\mathcal{A}$ of a ribbon.

Let C be a plane affine singular cubic curve given by the equation $y^2 = x^2(x+1)$ over a field k, $Q \in C$ is a closed point $x = y = 0$. We show that the sheaf $\mathcal{A} = \mathcal{O}_C((t))^Q$ is not coherent.

If it were coherent, then, by definition, for each $q \geq 1$ and $f_1, \dots f_q \in \mathcal{A}(U)$ the sheaf $\mathcal{K} = ker((\mathcal{A}|_U)^{\oplus q} \xrightarrow{(f_1,\dots,f_q)} (\mathcal{A}|_U))$ must be locally of finite type. We take $U \ni Q$, $q = 2$, and let f_1, f_2 be the images of x, y in $\mathcal{O}_C(U)$. Let $V \subset U$, $Q \in V$ be an open set such that $\mathcal{K}(V)$ is finitely generated in each point.

We consider an element $(b_1, b_2) \in \mathcal{K}(V)$ such that b_1, b_2 are the images of $-y, x$ in $\mathcal{O}_C(U)$. Then $(b_1, b_2) \in \mathcal{J}_Q((t))^{\oplus 2}(V)$, but $(b_1, b_2) \notin \mathcal{J}_Q^2((t))^{\oplus 2}(V)$. We note that elements $(b_1 t^m, b_2 t^m) \in \mathcal{K}(V)$ for any $m \in \mathbb{Z}$, and also satisfy this condition.

We note that for each $(a_1, a_2) \in \mathcal{K}(V)$ we have $a_i = \sum a_{ij} t^j$, where $a_{ij} \in \mathcal{J}_Q(V)$. Indeed, we must have $f_1 a_{1j} + f_2 a_{2j} = 0$ for all j, and this equality holds only if a_{ij} are polynomials in f_1, f_2 without free terms, i.e. belong to the ideal $\mathcal{J}_Q(V)$.

Let $g_1, \dots g_l$ be generators of $\mathcal{K}(V)$. Let they have orders (q_i, q_i'), $i = 1, \dots l$, where the order of an element $a \in \mathcal{A}(V)$ is equal to the degree (with respect to t) of the lowest term of a. For each $m \in \mathbb{Z}$ we must have

$$(b_1 t^m, b_2 t^m) = \sum_{i=1}^{l} w_{im} g_i \tag{4}$$

with $w_{im} \in \mathcal{A}(V)$. If $M = \min\{q_1, \dots, q_l, q_1' \dots, q_l'\}$, then all coefficients of t^j with $j < M$ on the right hand side of formula (4) must belong to $\mathcal{J}_Q^2(V)^{\oplus 2}$ for each m. But if $m \ll 0$ then there will be coefficients of t^j with $j < M$ on the left hand side of formula (4) that do not belong to $\mathcal{J}_Q^2(V)^{\oplus 2}$ (and the same is true for their images in the stalk of Q). We have a contradiction.

The same arguments show that the ideal $\mathcal{J}_Q(V)((t)) \subset \mathcal{A}(V)$ is not finitely generated, i.e. the ring $\mathcal{A}(V)$ is not Noetherian.

For convenience, we introduce also the following definition.

Definition 4. The sheaf of rings $\mathcal{F}$ on a topological space X is called weakly Noetherian, if there exists an open affine cover $\{U_\alpha\}_{\alpha \in I}$ such that $\mathcal{F}(U_\alpha)$ is a Noetherian ring for any $\alpha \in I$.

Example 3. This is an example of coherent, but not weakly Noetherian sheaf $\mathcal{A}$ of a ribbon.

We consider the ringed space $(C, \mathcal{A} = \mathcal{O}_C((t))^Q)$, where C is a reduced algebraic curve over a field k, $Q \in C$ is a smooth point. We will prove that the sheaf $\mathcal{A}$ is a coherent sheaf of rings. To prove that the sheaf $\mathcal{A}$ is a coherent sheaf of rings, it is enough to prove that the sheaf $\mathcal{K}$ from definition of coherence (see remark 2 above) is locally of finite type.

We consider an open $U \subset C$. If $U \not\ni Q$, then we have $(\mathcal{A}|_U)^{\oplus q} \simeq (\mathcal{O}_C((t))|_U)^{\oplus q}$ and therefore for any affine open subset $V \subset U$ the ring $(\mathcal{A}|_U)(V)$ is Noetherian. Clearly, $\mathcal{K}(V) = (\mathcal{K}'(V))_t$ and $\mathcal{A}(V) = (\mathcal{A}'(V))_t$, where

$$\mathcal{K}' = ker((\mathcal{A}'|_U)^{\oplus q} \stackrel{(f_1t^k, \dots, f_qt^k)}{\longrightarrow} (\mathcal{A}'|_U)), \quad \mathcal{A}' = \mathcal{O}_C[[t]]$$

for sufficiently large k (note that the definition of the sheaf $\mathcal{K}$ does not depend on changes $(f_1, \dots, f_q) \mapsto (f_1t^k, \dots, f_qt^k)$). The locally ringed space $(C, \mathcal{A}')$ is a Noetherian formal scheme (so, $\mathcal{A}'$ is a coherent sheaf, see [4, ch.I, §10.10]), therefore $\mathcal{K}'$ is locally of finite type, i.e. for each point $P \in U$ there exists open $V \subset U$, $P \in V$ and generators $(\beta_1, \dots, \beta_n)$ of $\mathcal{K}'(V)$ over $\mathcal{A}'(V)$ such that their images generate stalks $\mathcal{K}'_x$ for each $x \in V$. Clearly, $(\beta_1, \dots, \beta_n)$ are also generators of the $\mathcal{A}(V)$-module $\mathcal{K}(V)$, and they also generate stalks $\mathcal{K}_x$ over $\mathcal{A}_x$ for each $v \in V$, i.e. $\mathcal{K}$ is locally of finite type.

Let now $U \ni Q$, $f_1, \dots, f_q \in \mathcal{A}(U)$. Our sheaf $\mathcal{A}$ is a subsheaf of the sheaf $\tilde{\mathcal{A}} = \mathcal{O}_C((t))$, and the last sheaf is coherent, as we have proved above. We define the sheaf

$$\tilde{\mathcal{K}} = ker((\tilde{\mathcal{A}}|_U)^{\oplus q} \stackrel{(f_1, \dots, f_q)}{\longrightarrow} (\tilde{\mathcal{A}}|_U)).$$

It is locally of finite type, and $\mathcal{K}$ is the subsheaf of $\tilde{\mathcal{K}}$ as a sheaf of abelian groups.

For a given element $a = \sum\limits_j a_jt^j \in \tilde{\mathcal{A}}(V)$, $Q \in V$ we define its Q-order as follows:

$$\mathrm{ord}_Q(a) = \begin{cases} \min\{j \ : \ a_j \notin \mathcal{J}_Q\} \\ \infty, \quad \text{if for any} \quad j \quad a_j \in \mathcal{J}_Q. \end{cases}$$

Clearly, for any $a, b \in \mathcal{A}(V)$ we have

$$\mathrm{ord}_Q(ab) = \mathrm{ord}_Q(a) + \mathrm{ord}_Q(b).$$

For a given element $\alpha \in \tilde{\mathcal{A}}^{\oplus q}(V)$, $Q \in V$ we define its Q-order as a minimum of Q-orders of components of α, i.e.

$$\mathrm{ord}_Q(\alpha) = \min\{a_1, \ldots, a_q\} \quad \text{for} \quad \alpha = (a_1, \ldots, a_q).$$

Let $\alpha_1, \ldots, \alpha_k$ be generators of the $\tilde{\mathcal{A}}(V)$-module $\tilde{\mathcal{K}}(V)$, $V \ni Q$, such that their images generate stalks $\tilde{\mathcal{K}}_x$ for each $x \in V$. Without loss of generality we can assume that V is an affine open set, such that the maximal ideal of the point Q in $\mathcal{O}_C(V)$ is a principal ideal (y), $y \in \mathcal{O}_C(V)$. We can also assume that $\alpha_1, \ldots, \alpha_k \in \mathcal{K}(V)$, since otherwise we can replace them by $\alpha_1 t^{l_1}, \ldots \alpha_k t^{l_k}$. Since the maximal ideal of the point Q in $\mathcal{O}_C(V)$ is a principal ideal, we have $\alpha_i = y^{k_i}\alpha_i'$, where $k_i \geq 0$ and $\mathrm{ord}_Q(\alpha_i') < \infty$. We assume $\alpha_i' \in \mathcal{K}(V)$ again after multiplication them by some powers of t. Obviously, the elements $\alpha_i' \in \mathcal{K}(V)$ are also generators of $\tilde{\mathcal{K}}(V)$ and of stalks $\tilde{\mathcal{K}}_x$ for each $x \in V$. So, we can assume that $\mathrm{ord}_Q(\alpha_i) = 0$ for any $1 \leq i \leq k$.

Without loss of generality we can assume that the first component of α_1 be of zero Q-order. Since the ring $\mathcal{O}_C(V)$ has dimension 1, we can change $\alpha_1, \ldots, \alpha_k$ by $\alpha_1, \alpha_2 + x_2\alpha_1, \ldots, \alpha_k + x_k\alpha_1$ for some $x_2, \ldots, x_k \in \mathcal{A}(V)$ such that the first components of elements $\alpha_2 + x_2\alpha_1, \ldots, \alpha_k + x_k\alpha_1$ has infinite Q-order. If the Q-order of an element $\alpha_i + x_i\alpha_1$ is finite, we can again assume that it is zero, after multiplication him by an appropriate power of t.

The elements $\alpha_1, \alpha_2 + x_2\alpha_1, \ldots, \alpha_k + x_k\alpha_1$ are again generators of $\tilde{\mathcal{K}}(V)$ (and of $\tilde{\mathcal{K}}_x$ for each $x \in V$). They form a $k \times q$ matrix, whose entries lie in $\mathcal{A}(V)$ (the i-th row is the element $\alpha_i + x_i\alpha_1$). The corresponding $k \times q$ matrix of its Q-orders looks like

$$\begin{pmatrix} 0 & * & \ldots & * \\ \infty & * & \ldots & * \\ \vdots & \vdots & \ldots & \vdots \\ \infty & * & \ldots & * \end{pmatrix},$$

where some rows can consist only of infinities, and the minimal possible value in each row is zero.

If we permute some rows of our matrix, we don't change the system of generators of $\tilde{\mathcal{K}}(V)$ and of $\mathcal{K}_x$ for each $x \in V$. Therefore, we can assume that our matrix has the following property: the matrix of its Q-orders looks like

$$\begin{pmatrix} 0 & * & & & & \ldots & & & & & & * \\ \infty & \star & \ldots & \star & 0 & * & \ldots & & & & & * \\ \infty & \star & & & \ldots & & & \star & 0 & * & \ldots & * \\ \vdots & & & & & \ldots & & & & & \ddots & * \\ \infty & \infty & & & & \ldots & & & & & & \infty \\ \vdots & & & & & \ldots & & & & & & \\ \infty & \infty & & & & \ldots & & & & & & \infty \end{pmatrix},$$

where $\star > 0$. (The rows in the bottom of matrix contain only ∞.)

Clearly, the elementary transformations of rows like above lead to a new system of generators of $\tilde{\mathcal{K}}(V)$ (and of $\tilde{\mathcal{K}}_x$ for each $x \in V$).

So, repeating such elementary transformations and interchanges for the rows with zero Q-order, we will come to a system of generators $\alpha_1, \ldots, \alpha_k$ that satisfy the following additional property: for each $1 \le i \le k$ either $\mathrm{ord}_Q(\alpha_i) = \infty$, or $\mathrm{ord}_Q(\alpha_i) = 0$ and α_i has an l_i-component of zero Q-order such that the corresponding l_i-components of all other elements α_j, $j \neq i$ has infinite Q-order, see the matrix of Q-orders below:

$$\left(\begin{array}{cccccccccccc} 0 & * & \ldots & * & \infty & * & \ldots & * & \infty & * & \ldots & * \\ \infty & \star & \ldots & \star & 0 & * & \ldots & * & \infty & * & \ldots & * \\ \infty & \star & \ldots & \star & \infty & \star & \ldots & \star & 0 & * & \ldots & * \\ \vdots & \star & \cdots & \star & \infty & \star & \cdots & \star & \infty & \star & \ddots & * \\ \infty & \infty & & & & & \ldots & & & & & \infty \\ \vdots & & & & & & \ldots & & & & & \\ \infty & \infty & & & & & \ldots & & & & & \infty \end{array} \right).$$

Let $\alpha_1, \ldots, \alpha_l$ be of Q-order zero, and $\alpha_{l+1}, \ldots, \alpha_k$ be of Q-order ∞. Then $\alpha_j = y^{k_j}\alpha_j''$, $j \ge l+1$, where $\mathrm{ord}_Q(\alpha_j'') < \infty$. After multiplication of α_j'' by some power of t we have $\alpha_j = y^{k_j}t^{m_j}\alpha_j'$, $j \ge l+1$ for some $k_j > 0$ and some m_j such that $\mathrm{ord}_Q(\alpha_j') = 0$.

We claim that the elements $\alpha_1, \ldots, \alpha_l, \alpha_{l+1}', \ldots, \alpha_k'$ are generators of the $\mathcal{A}(V)$-module $\mathcal{K}(V)$ such that their images generate stalks $\mathcal{K}_x$ for any $x \in V$.

Indeed, if $x \in V$, $x \neq Q$, then it is clear by the choice of elements $\alpha_1, \ldots, \alpha_k$ in the beginning, because $\mathcal{K}_x = \tilde{\mathcal{K}}_x$. Now let $b \in \mathcal{K}_Q$. Then $b = \sum b_j\alpha_j$ for some $b_j \in \tilde{\mathcal{A}}_Q$. We have $b_j\alpha_j \in \mathcal{K}_Q$ for all $j \ge l+1$, since $\mathrm{ord}_Q(b_j\alpha_j) = \infty$. The first component of α_1 is of zero Q-order, and the Q-orders of the first components of all other α_i, $i \ge 2$ are infinite. Since $b \in \mathcal{K}_Q$, the Q-order of the first component of $b_1\alpha_1$ must be therefore greater or equal to zero. Hence, $\mathrm{ord}_Q(b_1) \ge 0$ and $b_1 \in \mathcal{A}_Q$. Analogously, $b_j \in \mathcal{A}_Q$ for $j \le l$. Now for $j \ge l+1$ we have $b_j\alpha_j = b_jy^{k_j}t^{m_j}\alpha_j'$ with $k_j > 0$, and $b_j' := b_jy^{k_j}t^{m_j} \in \mathcal{A}_Q$ because $k_j > 0$. So,

$$b = \sum_{i=1}^{l} b_i\alpha_i + \sum_{j=l+1}^{k} b_j'\alpha_j',$$

where $b_i, b_j' \in \mathcal{A}_Q$, and we are done.

Nevertheless, the sheaf $\mathcal{A}$ is not weakly Noetherian. For example, consider the following infinite increasing system of ideals in $\mathcal{A}(U)$ (for any $U \ni Q$):

$$J_k := \{c = \sum_{i=l}^{\infty} c_it^i, \text{ where } c_i \in \mathcal{J}_Q(U) \text{ and } c_i \in \mathcal{J}_Q^2(U) \text{ for } i < -k \ \}.$$

Clearly, $J_1 \subset J_2 \subset \ldots$ does not stabilize.

Remark 3. The situation described in example 3 is similar to the situation of rank 2 valuation ring $\mathcal{O}' = k[[t]] + uk((t))[[u]]$ in 2-dimensional local field $k((t))((u))$. The ring

$\mathcal{O}'$ is also non Noetherian (see [13]), but one can prove that the ring $\mathcal{O}'$ is coherent by the same methods as above.

Example 4. Now let's consider one more example. Let C be a reduced algebraic curve over a field k. Consider a ringed space $(C, \mathcal{A})$, where

$$\mathcal{A} = \{\sum_{j=N}^{\infty} \mathcal{O}_C \cdot t_j, \quad t_0 = 1, \quad t_i t_j = 0 \text{ for all } i, j \neq 0\}$$

Clearly, $\mathcal{A}$ is a sheaf that satisfies all conditions of definition 1. So, $(C, \mathcal{A})$ is a ribbon.

Obviously, the sheaf $\mathcal{A}$ is also not coherent and not weakly Noetherian. Moreover, $\mathcal{A}_0$ is not coherent. To see this, it is enough to consider the kernel of multiplication by t_1. Clearly, this kernel can not be locally of finite type.

Under certain conditions on the sheaf $\mathcal{A}$ of a ribbon we can prove in the following lemma that it will be coherent, as well as any torsion free sheaf of finite rank on this ribbon will be coherent. (We will define torsion free sheaves later, see definition 11 and remark 10).

Definition 5. We will say that the sheaf $\mathcal{A}$ of a ribbon $(C, \mathcal{A})$ satisfies (*), if the following condition holds:

$$\begin{gathered}\text{there is an affine open cover } \{U_\alpha\}_{\alpha\in I} \text{ of } C \text{ such that for any } \alpha \in I \\ \text{there is } k > 0 \text{ and an invertible section } a \in \mathcal{A}_k(U_\alpha) \subset \mathcal{A}(U_\alpha).\end{gathered} \tag{*}$$

Definition 6. For an open set U we define the function of order ord_U on $\mathcal{A}(U)$ in the following way: if an element $b \in \mathcal{A}_l(U) \backslash \mathcal{A}_{l+1}(U)$, then $\mathrm{ord}_U(b) = l$. Sometimes, if it is clear from the context, we will omit the index U.

Now we prove the following lemma.

Lemma 2. *Let the sheaf $\mathcal{A}$ of a ribbon $(C, \mathcal{A})$ satisfy (*). Then it is weakly Noetherian and coherent. Moreover, for any affine open subset U of C the ring $\mathcal{A}(U)$ is a Noetherian ring.*

Proof. Let $\{U_\alpha\}$ be the cover from (*). For an open $U_\alpha \subset C$ let $a \in \mathcal{A}^*(U_\alpha)$, $a \in \mathcal{A}_k(U_\alpha)$, $k > 0$. From the definition of ribbon (definition 1) it follows that $a^{-1} \in \mathcal{A}_l(U_\alpha) \backslash \mathcal{A}_{l+1}(U_\alpha)$, where $l \le -k$. Clearly, $\mathcal{A}(U_\alpha) = \mathcal{A}_0(U_\alpha)_a$. By propositions 1 and 3, the ring $\mathcal{A}_0(U_\alpha)/\mathcal{A}_{-l}(U_\alpha)$ is Noetherian.

Let $\tilde{I} \subset \mathcal{A}(U_\alpha)$ be an ideal. Let $I = \tilde{I} \cap \mathcal{A}_0(U_\alpha)$. Set $I_{-l} = I/I \cap \mathcal{A}_{-l}(U_\alpha)$. Let $\bar{g}_1, \ldots, \bar{g}_s$ be generators of I_{-l} in $\mathcal{A}_0(U_\alpha)/\mathcal{A}_{-l}(U_\alpha)$, and $g_1, \ldots, g_s$ be any their representatives in I. Let $x \in I$ be any element, $x \in \mathcal{A}_j(U_\alpha) \backslash \mathcal{A}_{j+1}(U_\alpha)$. If $j < -l$, then there are $b_1, \ldots, b_s \in \mathcal{A}_0(U_\alpha)$ such that $x - \sum_m b_m g_m \in \mathcal{A}_i(U_\alpha) \backslash \mathcal{A}_{i+1}(U_\alpha)$ with $i \ge -l$. If $j \ge -l$, then $a^{-1}x \in I$, and for some $m \ge 1$ we have $a^{-m}x \in \mathcal{A}_i(U_\alpha) \backslash \mathcal{A}_{i+1}(U_\alpha)$ with $0 \le i < -l$. We iterate this procedure. Since $\mathrm{ord}(a) > 0$, and $\mathcal{A}_0(U_\alpha)$ is a complete

and Hausdorff space, we can deduce that $g_1, \dots, g_s$ generate $\tilde{I}$, hence I. So, $\mathcal{A}(U_\alpha)$ is a Noetherian ring.

Analogously we can show that $\mathcal{A}_0(U_\alpha)$ is also a Noetherian ring. Namely, for an ideal $J \subset \mathcal{A}_0(U_\alpha)$ let $\tilde{J}$ be an ideal generated by J in $\mathcal{A}(U_\alpha)$. If $\tilde{J} = (1)$, then $a^r \in J$ for some $r > 0$. For any $i \ge -lr$ we have $(a^r) \supseteq \mathcal{A}_i(U_\alpha)$. Therefore, elements $g_1, \dots, g_s$, whose images in $\mathcal{A}_0(U_\alpha)/\mathcal{A}_{-lr}(U_\alpha)$ generate the ideal $J/J \cap \mathcal{A}_{-lr}(U_\alpha)$, and the element a^r will generate the ideal J.

If $\tilde{J} \ne (1)$, then $\tilde{J} = (g_1, \dots, g_s)$ as above, where $g_1, \dots, g_s \in \mathcal{A}_0(U_\alpha)$. As it was shown above, for any sufficiently large i an element $x \in J \cap \mathcal{A}_i(U_\alpha)$ can be written as $x = a^h \sum_m b_m g_m$ with $b_1, \dots, b_s \in \mathcal{A}_0(U_\alpha)$, $h > 0$. On the other hand, for a sufficiently large h we have $a^h g_1, \dots, a^h g_s \in J$. Therefore, there exists a natural N such that for any $x \in J \cap \mathcal{A}_i(U_\alpha)$ with $i > N$ we have $x \in (a^h g_1, \dots, a^h g_s) \subset \mathcal{A}_0(U_\alpha)$. Now, if $g_1', \dots, g_t' \in J$ are representatives of generators of the ideal $J/J \cap \mathcal{A}_N(U_\alpha)$, then the system $g_1', \dots, g_t', a^h g_1, \dots, a^h g_s$ is a system of generators of the ideal J.

To show that $\mathcal{A}$ is coherent, it is enough to prove that the sheaf $\mathcal{K}$ from the definition of a coherent sheaf (see remark 2) is locally of finite type for each U_α.

For any open $V \subset U_\alpha$ we have $\mathcal{K}(V) = (\mathcal{K}'(V))_a$ and $\mathcal{A}(V) = (\mathcal{A}_0(V))_a$, where

$$\mathcal{K}' = ker((\mathcal{A}_0|_{U_\alpha})^{\oplus q} \stackrel{(f_1 a^k, \dots, f_q a^k)}{\longrightarrow} (\mathcal{A}_0|_{U_\alpha}))$$

for sufficiently large k (as in example 3). We also have

$$\varprojlim_{n \ge 0} \mathcal{A}_0(V)/a^n \mathcal{A}_0(V) = \mathcal{A}_0(V),$$

because for ideal $(a) = a\mathcal{A}_0(V)$ we always have $\mathcal{A}_n \supseteq (a)^n \supseteq \mathcal{A}_i(V)$ for $i \ge -ln$ and for any n, and $(a)^n \supseteq \mathcal{A}_i(V) \supseteq (a)^i$ for $n \le [i/(-l)]$.

Combining all together, we obtain that the following locally ringed spaces are isomorphic:

$$(U_\alpha, \mathcal{A}_0|_{U_\alpha}) \simeq (\widehat{\operatorname{Spec} \mathcal{A}_0(U_\alpha)})_Y,$$

where Y is a closed subscheme of $\operatorname{Spec} \mathcal{A}_0(U_\alpha)$ given by the ideal (a), and the formal Noetherian scheme $(\widehat{\operatorname{Spec} \mathcal{A}_0(U_\alpha)})_Y$ is a completion of the scheme $\operatorname{Spec} \mathcal{A}_0(U_\alpha)$ along Y. So, by [4, ch.I, §10.10], the sheaf $\mathcal{A}_0|_{U_\alpha}$ is coherent, and the sheaf $\mathcal{K}'$ of $\mathcal{A}_0|_{U_\alpha}$-modules is locally of finite type. Therefore the sheaf $\mathcal{K}$ of $\mathcal{A}|_{U_\alpha}$-modules is locally of finite type.

We show the last property of the lemma. At first, we note that for any open $V \subset U_\alpha$ the ring $\mathcal{A}(V)$ satisfies (*) and therefore is Noetherian, as it was shown above. Since for an open affine $U = \underline{\operatorname{Spec}} B$ there is a base of topology consisting of open sets $D(f) \simeq \underline{\operatorname{Spec}} B_f$ and any affine set is quasicompact, we can cover the set U by finite number of affine open sets $U_i \simeq \underline{\operatorname{Spec}} B_{f_i}$ such that the rings $\mathcal{A}(U_i)$ satisfy (*) and are Noetherian. By definition of a ribbon and by proposition 3, we can take $B = \mathcal{A}_0(U)/\mathcal{A}_1(U)$, and $f_i \in \mathcal{A}_0(U)/\mathcal{A}_1(U)$, f_i generate the ideal (1) of the ring B.

Now we prove the following statement. Let $I \subset \mathcal{A}(U)$ be an ideal and $\phi_i : \mathcal{A}(U) \to \mathcal{A}(U_i)$ are the restriction homomorphisms, $i = 1, \dots, r$. Then

$$I = \bigcap_i \phi_i^{-1}(\phi_i(I) \cdot \mathcal{A}(U_i)).$$

Obviously, we have $I \subset \bigcap_i \phi_i^{-1}(\phi_i(I) \cdot \mathcal{A}(U_i))$. Now let

$$b \in \bigcap_i \phi_i^{-1}(\phi_i(I) \cdot \mathcal{A}(U_i)) \quad , \quad b \in \mathcal{A}_k(U) \backslash \mathcal{A}_{k+1}(U).$$

Let

$$\phi_i(b) = \sum_{j=1}^{r_i} \phi_i(a_j) g_j,$$

where $g_j \in \mathcal{A}(U_i)$, $a_j \in I$. We have $\phi_i(b) \in \mathcal{A}_k(U_i)$ and therefore

$$\phi_i(b) \mod \mathcal{A}_{k+1}(U_i) \;=\; \sum_{j=1}^{r_i} (\phi_i(a_j) \mod \mathcal{A}_{k+1-\mathrm{ord}(g_j)}(U_i))(g_j \mod \mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U_i)).$$

We consider the homomorphisms

$$\bar{\phi}_i^j \;:\; \mathcal{A}_{\mathrm{ord}(g_j)}(U)/\mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U) \longrightarrow \mathcal{A}_{\mathrm{ord}(g_j)}(U_i)/\mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U_i),$$

which are induced by ϕ_i. By proposition 1, the sheaf $\mathcal{A}_{\mathrm{ord}(g_j)}/\mathcal{A}_{k+1-\mathrm{ord}(a_j)}$ is a coherent sheaf on the scheme $X_l = (C, \mathcal{A}_0/\mathcal{A}_{l+1})$, where $l = k - \mathrm{ord}(a_j) - \mathrm{ord}(g_j)$ (we assume that $l \ge 0$, since otherwise our sheaf is trivial and there is nothing to prove). Therefore, $\bar{\phi}_i^j$ is a localization map, and for any element $x \in \mathcal{A}_{\mathrm{ord}(g_j)}(U_i)/\mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U_i)$ there exists a natural n such that $f_{ij}^n x = \bar{\phi}_i^j(\tilde{x})$, where $\tilde{x} \in \mathcal{A}_{\mathrm{ord}(g_j)}(U)/\mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U)$, $f_{ij} \in \mathcal{A}_0(U)/\mathcal{A}_{l+1}(U)$ and $f_{ij} \mod \mathcal{A}_1(U) = f_i$ (see [6, lemma 5.3]). Note that we can choose $f_{ij} = \tilde{f}_i \mod \mathcal{A}_{l+1}(U)$, where $\tilde{f}_i$ is a fixed representative of f_i in $\mathcal{A}_0(U)$, for all j. Hence there exists a natural N such that

$$\phi_i(\tilde{f}_i^N)\phi_i(b) \mod \mathcal{A}_{k+1}(U_i) \;=\; \sum_{j=1}^{r_i} (\phi_i(a_j g_j') \mod \mathcal{A}_{k+1}(U_i)),$$

where $g_j' \in \mathcal{A}(U)$ and

$$\phi_i(g_j') \mod \mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U_i) \;=\; \phi_i(\tilde{f}_i^N) g_j \mod \mathcal{A}_{k+1-\mathrm{ord}(a_j)}(U_i).$$

Let k' be an integer such that $a_j g_j' \in \mathcal{A}_{k'}(U)$ for any j (note that $k' \le k$). Then, repeating the arguments above to the coherent sheaf $\mathcal{A}_{k'}/\mathcal{A}_{k+1}$ we obtain that there exists a natural M such that

$$\tilde{f}_i^M b \mod \mathcal{A}_{k+1}(U) \;\in\; I \mod \mathcal{A}_{k+1}(U).$$

Note that we can choose M unique for all i and that the elements $\tilde{f}_i^M$ generate the ideal (1) in $\mathcal{A}_0(U)$, i.e. $\sum_i c_i \tilde{f}_i^M = 1$ for some $c_i \in \mathcal{A}_0(U)$. Therefore,

$$b \mod \mathcal{A}_{k+1}(U) \;=\; \sum c_i \tilde{f}_i^M b \mod \mathcal{A}_{k+1}(U) \;\in\; I \mod \mathcal{A}_{k+1}(U).$$

So, there exists $b_1 \in I$, $b_1 \in \mathcal{A}_k(U)$ such that $(b - b_1) \in \bigcap_i \phi_i^{-1}(\phi_i(I) \cdot \mathcal{A}(U_i))$ and $\mathrm{ord}(b - b_1) > \mathrm{ord}(b)$. We repeat the arguments above for the element $b - b_1$ and so on. Since the ring $\mathcal{A}(U)$ has complete and Hausdorff topology, we obtain that $b \in I$.

Now it is easy to show that the ring $\mathcal{A}(U)$ is Noetherian. Let $I_1 \subset I_2 \subset \dots$ be an increasing chain of ideals in $\mathcal{A}(U)$. Then for each i the chain

$$\phi_i(I_1) \cdot \mathcal{A}(U_i) \subset \phi_i(I_2) \cdot \mathcal{A}(U_i) \subset \dots$$

is stable, since $\mathcal{A}(U_i)$ is a Noetherian ring. Since there are only finite number of i, the first chain is also stable, where from $\mathcal{A}(U)$ is a Noetherian ring.

□

Definition 7. A ribbon $(C, \mathcal{A})$ is called algebraizable if it is locally isomorphic on C to a ribbon from example 1.

Example 5. The sheaf $\mathcal{A} = \mathcal{O}_{\hat{X}_C}(*C)$ with the filtration $\mathcal{A}_i = \mathcal{O}_{\hat{X}_C}(-iC)$ on a surface X with an effective reduced Cartier divisor C from example 1 satisfies the conditions of lemma 2. Indeed, the local equation of C in X is an invertible element that belong to $\mathcal{A}_1(U)$, and its inverse belongs to $\mathcal{A}_{-1}(U)$.

In particular, it follows that the ribbons from example 2, example 3 and example 4 are not algebraizable, because they are not weakly Noetherean.

Remark 4. Structure sheaves of algebraizable ribbons satisfy more pretty property, which is useful in studying of the Picard group of a ribbon, see proposition 9 below.

Example 6. We consider an example of a ribbon with weakly Noetherian and coherent structure sheaf $\mathcal{A}$, but which is not algebraizable. It can be constructed in the same way as in example 4.

Let C be a reduced algebraic curve over a field k. Consider a ringed space $(C, \mathcal{A})$, where

$$\mathcal{A} = \{\sum_{j=N}^{\infty} \mathcal{O}_C \cdot t_j, \quad t_0 = 1, \quad t_{2i} = t_2^i, \quad t_{2i+1} = t_1 t_2^i, \quad t_1^2 = 0\}.$$

Clearly, $\mathcal{A}$ is a sheaf that satisfies all conditions of definition 1. So, $(C, \mathcal{A})$ is a ribbon. By lemma 2 $\mathcal{A}$ is a weakly Noetherian and coherent sheaf (since t_2 is an invertible section of $\mathcal{A}(U)$ for any open $U \subset C$). But $(C, \mathcal{A})$ is not algebraizable, because if it were algebraizable, there should be an open affine cover of C such that for any open U from this cover there exists an invertible element a that belong to $\mathcal{A}_1(U) \backslash \mathcal{A}_2(U)$ and $a^{-1} \in \mathcal{A}_{-1}(U) \backslash \mathcal{A}_0(U)$. Obviously, there are no such sections in $\mathcal{A}(U)$ for any U.

3.3 Analytic ribbons

When a ground field is $\mathbf{C}$, we can also work in the analytic category to define ribbons over $\mathbf{C}$, replacing "algebraic coherent" by "analytic coherent sheaf" (for $\mathcal{A}_i/\mathcal{A}_{i+1}$, $i \in \mathbb{Z}$) in definition 1. Then we obtain the notion of an analytic ribbon $(C, \mathcal{A})$.

We define an analytic ind-pro-coherent sheaf $\mathcal{F}$ on analytic ribbon $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ as a filtered sheaf of $\mathcal{A}$-modules (with a descending filtration by subsheaves) satisfying properties 1, 3, 4 and the property

$2'$. $\mathcal{F}_j/\mathcal{F}_{j+1}$ is an analytic coherent sheaf on C for any $j \in \mathbb{Z}$

instead of property 2 of definition 3.

Remark 5. Since the underlying topological space is non-Noetherian in this case, we have to take the sheaf $\mathcal{F}$ associated with the presheaf $\mathcal{F}' : V \mapsto \varinjlim_i \mathcal{F}_i(V)$.

We have the following proposition (compare with proposition 3).

Proposition 4. *We have the following properties for an analytic ind-pro-coherent sheaf $\mathcal{F}$ on an analytic ribbon $\overset{\circ}{\mathrm{X}}_\infty = (C, \mathcal{A})$, where C is an irreducible complex algebraic curve.*

1. *$\mathcal{F}_i/\mathcal{F}_{i+j+1}$ is an analytic coherent sheaf on X_j for any $j \geq 0$, $i \in \mathbb{Z}$.*
2. *$\mathcal{F}_i(U)/\mathcal{F}_j(U) \to (\mathcal{F}_i/\mathcal{F}_j)(U)$ is an isomorphism for $i < j$ and for Stein open sets $U \subset C$.*
3. *$H^q(U, \mathcal{F}_i) = 0$ for any Stein open subset $U \subset C$ and $q > 0$, $i \in \mathbb{Z}$.*

Remark 6. We note that every complex analytic space of dimension 1, which has no compact irreducible components, is a Stein space (see, for example, [11]).

Proof. The proof of statement 1 and statement 2 of this proposition is the same as in proposition 3. (We use that for any analytic coherent sheaf $\mathcal{G}$ on a Stein space U we have $H^q(U, \mathcal{G}) = 0$ for $q > 0$.)

Now we prove statement 3 of the proposition. By remark 6 we have that any open subset V of a Stein subset $U \subset C$ is a Stein space again. Therefore, if $\{U_\alpha\}$ is an open covering of U, then every open U_α is a Stein space. Let $\check{\mathcal{C}}^\bullet(\{U_\alpha\}, \mathcal{F}_i)$ be the Čech-complex of this covering for the sheaf $\mathcal{F}_i$. We obtain that

$$\check{\mathcal{C}}^\bullet(\{U_\alpha\}, \mathcal{F}_i) = \varprojlim_{j>i} \check{\mathcal{C}}^\bullet(\{U_\alpha\}, \mathcal{F}_i/\mathcal{F}_j).$$

We consider the following natural complex $D_i^\bullet$ for any $i \in \mathbb{Z}$:

$$0 \longrightarrow \mathcal{F}_i(U) \longrightarrow \check{\mathcal{C}}^0(\{U_\alpha\}, \mathcal{F}_i) \longrightarrow \check{\mathcal{C}}^1(\{U_\alpha\}, \mathcal{F}_i) \longrightarrow \dots,$$

i.e. $D_i^n = 0$ for $n < -1$, $D_i^{-1} = \mathcal{F}_i(U)$, and $D_i^n = \check{\mathcal{C}}^n(\{U_\alpha\}, \mathcal{F}_i)$ for $n \geq 0$.

We have that for any $i \in \mathbb{Z}$

$$D_i^\bullet = \varprojlim_{j>i} D_{i,j}^\bullet,$$

where the complex $D_{i,j}^\bullet$ is defined in the following natural way for any $j \geq i \in \mathbb{Z}$: $D_{i,j}^n = 0$ for $n < -1$, $D_{i,j}^{-1} = (\mathcal{F}_i/\mathcal{F}_j)(U)$, and $D_{i,j}^n = \check{\mathcal{C}}^n(\{U_\alpha\}, \mathcal{F}_i/\mathcal{F}_j)$ for $n \geq 0$.

From statement 2 of this proposition we have that for any fixed $i \in \mathbb{Z}$, for any $n \in \mathbb{Z}$ the projective system $(D_{i,j}^n, j \geq i)$ satisfies the ML-condition, because the maps in this projective system are surjective maps.

For any $j \geq i \in \mathbb{Z}$ the complex $D_{i,j}^\bullet$ is an acyclic complex, because the Čech cohomology

$$\check{H}^0(\{U_\alpha\}, \mathcal{F}_i/\mathcal{F}_j) = (\mathcal{F}_i/\mathcal{F}_j)(U),$$
$$\check{H}^n(\{U_\alpha\}, \mathcal{F}_i/\mathcal{F}_j) = H^n(U, \mathcal{F}_i/\mathcal{F}_j) = 0 \quad \text{for any} \quad n > 0.$$

Therefore for any $i \in \mathbb{Z}$ the complex $D_i^\bullet$ is an acyclic complex, as it follows from the following lemma.

Lemma 3. *Let $(K_l^\bullet, l \geq 0)$ be a projective system of acyclic complexes $K_l^\bullet$ of abelian groups. We suppose that for any $n \in \mathbb{Z}$ the projective system $(K_l^n, l \geq 0)$ satisfies the ML-condition. Then the complex*

$$K^\bullet = \varprojlim_{l \geq 0} K_l^\bullet$$

is an acyclic complex.

Proof of the lemma. Let maps $d_l^n : K_l^n \to K_l^{n+1}$, $n \in \mathbb{Z}$ be the differentials in complex $K_l^\bullet$, $l \geq 0$. We have the following exact sequences:

$$0 \longrightarrow \operatorname{Ker} d_l^n \longrightarrow K_l^n \longrightarrow \operatorname{Im} d_l^n \longrightarrow 0. \tag{5}$$

Since the complex $K_l^\bullet$ is an acyclic complex, we have that $\operatorname{Im} d_l^n = \operatorname{Ker} d_l^{n+1}$ for any n.

Since for any n the projective system $(K_l^n, l \geq 0)$ satisfies ML-condition, we obtain from exact sequence (5) that for any n the projective system $(\operatorname{Im} d_l^{n-1}, l \geq 0) = (\operatorname{Ker} d^{n+1}, l \geq 0)$ satisfies ML-condition. Let maps $d^n : K^n \to K^{n+1}$ be the differentials in complex $K^\bullet$. Now, using lemma 1 and that always $\operatorname{Ker} d_n = \varprojlim_{n \geq 0} \operatorname{Ker} d_n^l$ for any $n \in \mathbb{Z}$, we obtain that the projective limit with respect to $l \geq 0$ of sequences (5) will give the following exact sequence for any $n \in \mathbb{Z}$:

$$0 \longrightarrow \operatorname{Ker} d^n \longrightarrow K^n \longrightarrow \operatorname{Im} d^n \longrightarrow 0.$$

Therefore, for any $n \in \mathbb{Z}$ we have

$$\operatorname{Im} d^n = \varprojlim_{l \geq 0} \operatorname{Im} d_l^n = \varprojlim_{l \geq 0} \operatorname{Ker} d_l^{n+1} = \operatorname{Ker} d^{n+1}.$$

Therefore the complex $K^\bullet$ is an acyclic complex. The lemma is proved.

$\square$

Now we finish the proof of proposition 4. We have proved that $\check{H}^q(\{U_\alpha\}, \mathcal{F}_i) = 0$ for any $i \in \mathbb{Z}$ and any $q > 0$. Therefore $\check{H}^q(U, \mathcal{F}_i) = 0$ for any $i \in \mathbb{Z}$ and any $q > 0$. Hence, for any $i \in \mathbb{Z}$

$$H^1(U, \mathcal{F}_i) = \check{H}^1(U, \mathcal{F}_i) = 0.$$

Furthermore, we have a spectral sequence with initial term

$$E_2^{pq} = \check{H}^p(U, \mathcal{H}^q(\mathcal{F}_i)) \Longrightarrow H^{p+q}(U, \mathcal{F}_i), \tag{6}$$

where $\mathcal{H}^q(\mathcal{F}_i)$ is the presheaf $V \subset U \mapsto \mathcal{H}^q(\mathcal{F}_i)(V) = H^q(V, \mathcal{F}_i)$ (see [3]). So, since any open subset $V \subset U$ is a Stein subspace again, we have in our situation

$$H^2(U, \mathcal{F}_i) = \check{H}^0(U, \mathcal{H}^2(\mathcal{F}_i)).$$

To obtain that $H^2(U, \mathcal{F}_i) = 0$ it is sufficient to show that for any point $x \in C$

$$\varinjlim_{x \in V \subset C} H^2(V, \mathcal{F}_i) = 0.$$

It follows from the following fact ([3, lemma 3.8.2]: for any point $x \in C$, for any $p > 0$

$$\lim_{\substack{\longrightarrow \\ x \in V \subset C}} H^p(V, \mathcal{F}_i) = 0 \tag{7}$$

Now by induction on q, by the same methods as for $q = 2$, using spectral sequence (6) and equality (7), we obtain that $H^q(U, \mathcal{F}_i) = 0$ for all $q > 0$. The proposition is proved.

□

Corollary 2. *Let* $\overset{\circ}{\mathrm{X}}_\infty = (C, \mathcal{A})$ *be an analytic ribbon. Let* $\mathcal{F}$ *be an analytic ind-pro-coherent sheaf on* $\overset{\circ}{\mathrm{X}}_\infty$, *and* C *be an irreducible compact space.*

1. *If* $C = U_1 \cup U_2$, *where* U_1 *and* U_2 *are Stein open subsets, then we have an exact sequence for any* $i \in \mathbb{Z}$

$$0 \to H^0(C, \mathcal{F}_i) \to H^0(U_1, \mathcal{F}_i) \oplus H^0(U_2, \mathcal{F}_i) \to H^0(U_1 \cap U_2, \mathcal{F}_i) \to H^1(C, \mathcal{F}_i) \to 0.$$

2. $H^*(C, \mathcal{F}_i) = \varprojlim_{j > i} H^*(C, \mathcal{F}_i / \mathcal{F}_j)$, $i \in \mathbb{Z}$.

3. $H^q(C, \mathcal{F}_i) = 0$ *for* $q > 1$, $i \in \mathbb{Z}$.

Proof is similar to the proof of corollary 1 of proposition 3. We have to use the following Mayer-Vietoris exact sequence for a sheaf $\mathcal{G}$ on C:

$$\ldots \longrightarrow H^{k-1}(U_1 \cap U_2, \mathcal{G}) \longrightarrow H^k(C, \mathcal{G}) \longrightarrow H^k(U_1, \mathcal{G}) \oplus H^k(U_2, \mathcal{G}) \longrightarrow \ldots$$

□

4 The Picard group of a ribbon

We recall that for a ringed space $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ the Picard group $Pic(\overset{\circ}{X}_\infty) = H^1(C, \mathcal{A}^*)$, and for the ringed space $X_\infty = (C, \mathcal{A}_0)$ also the Picard group $Pic(X_\infty) = H^1(C, \mathcal{A}_0^*)$.

Proposition 5. *Let* $\overset{\circ}{\mathrm{X}}_\infty = (C, \mathcal{A})$ *be a ribbon over an Artinian ring* A. *We suppose that* C *is either projective, or affine curve over* $\operatorname{Spec} A$. *Then*

$$Pic(X_\infty) = \varprojlim_{i \geq 0} Pic(X_i).$$

Proof. We denote for any $j \geq i \geq 0$ the following sheaves $\mathcal{G}_{i,j} = \frac{1 + \mathcal{A}_{i+1}}{1 + \mathcal{A}_{j+1}}$ on C. Then we have the following exact sequences:

$$1 \longrightarrow \mathcal{G}_{i,j} \longrightarrow \mathcal{O}^*_{X_j} \longrightarrow \mathcal{O}^*_{X_i} \longrightarrow 1. \tag{8}$$

We denote for any $i \geq 0$ the following sheaf $\mathcal{G}_i = 1 + \mathcal{A}_{i+1} \subset \mathcal{A}_0^*$ on C. Then we have

$$\mathcal{G}_i = \varprojlim_{j \geq i} \mathcal{G}_{i,j}.$$

For any $j \geq i \geq 0$ we have the following exact sequences:

$$1 \longrightarrow \mathcal{G}_{j,j+1} \longrightarrow \mathcal{G}_{i,j+1} \longrightarrow \mathcal{G}_{i,j} \longrightarrow 1. \tag{9}$$

For any $j \geq 0$ we have $\mathcal{G}_{j,j+1} \simeq \mathcal{A}_{j+1}/\mathcal{A}_{j+2}$. Therefore from sequence (9) we obtain that for any affine open subset $U \subset C$ the maps $H^0(U, \mathcal{G}_{i,j+1}) \to H^0(U, \mathcal{G}_{i,j})$ are surjective for any $j \geq i \geq 0$, by induction on j we obtain that $H^1(U, \mathcal{G}_{i,j}) = 0$ for any $j \geq i \geq 0$. Therefore, arguing as in the proof of assertion 3 of proposition 3, we obtain that $H^1(U, \mathcal{G}_i) = 0$ for any $i \geq 1$.

Since C is a curve over an Artinian ring, there are some affine open subsets U_1 and U_2 of C such that $C = U_1 \cup U_2$. Therefore the following Mayer-Vietoris sequence is exact:

$$0 \to H^0(C, \mathcal{G}_i) \to H^0(U_1, \mathcal{G}_i) \oplus H^0(U_2, \mathcal{G}_i) \to H^0(U_1 \cap U_2, \mathcal{G}_i) \to H^1(C, \mathcal{G}_i) \to 0. \tag{10}$$

Also for any $j \geq i \geq 0$ we have the following exact sequences:

$$0 \to H^0(C, \mathcal{G}_{i,j}) \to H^0(U_1, \mathcal{G}_{i,j}) \oplus H^0(U_2, \mathcal{G}_{i,j}) \to H^0(U_1 \cap U_2, \mathcal{G}_{i,j}) \to H^1(C, \mathcal{G}_{i,j}) \to 0. \tag{11}$$

We note that for any fixed $i \geq 0$ the projective system $(H^0(U_1, \mathcal{G}_{i,j}) \oplus H^0(U_2, \mathcal{G}_{i,j}), j \geq i)$ satisfies the ML-condition, because the maps in this system are surjective. By the same reason, if the curve C is affine, then for any fixed $i \geq 0$ the projective system $(H^0(C, \mathcal{G}_{i,j}), j \geq i)$ satisfies the ML-condition. If the curve C is projective, then we consider the following exact sequences which follow from sequences (8):

$$0 \longrightarrow H^0(C, \mathcal{G}_{i,j}) \longrightarrow H^0(C, \mathcal{O}^*_{X_j}) \longrightarrow H^0(C, \mathcal{O}^*_{X_i}). \tag{12}$$

We have that A-modules $(H^0(C, \mathcal{O}_{X_j}), j \geq 0)$ satisfy the ML-condition, because they are Artinian A-modules. Therefore the groups $H^0(C, \mathcal{O}^*_{X_j}) = H^0(C, \mathcal{O}_{X_j})^*$ satisfy the ML-condition as invertible elements of the corresponding algebras for which: 1) we have ML-condition and 2) maps in projective system have nilpotent kernels. Whence, for the fixed $i \geq 0$ from exact sequence (12) we obtain that the projective system $(H^0(C, \mathcal{G}_{i,j}), j \geq i)$ satisfies the ML-condition as the kernels of the maps to the constant group $H^0(C, \mathcal{O}^*_{X_i})$.

Now we apply lemma 1 to obtain that for the fixed $i \geq 0$ exact sequence (10) is the projective limit of exact sequences (11) with respect to $j \geq i$. Therefore we have

$$H^1(C, \mathcal{G}_i) = \varprojlim_{j \geq i} H^1(C, \mathcal{G}_{i,j}).$$

Let $i = 0$, then from exact sequence (8) we obtain the following exact sequence for $j \geq 0$:

$$0 \longrightarrow H^0(C, \mathcal{G}_{0,j}) \longrightarrow H^0(C, \mathcal{O}^*_{X_j}) \longrightarrow H^0(C, \mathcal{O}^*_C) \longrightarrow$$

$$\longrightarrow H^1(C, \mathcal{G}_{0,j}) \longrightarrow H^1(C, \mathcal{O}^*_{X_j}) \longrightarrow H^1(C, \mathcal{O}^*_C) \longrightarrow 0.$$

Every term of this sequence satisfies the ML-condition with respect to j. (For the zero cohomology it is proved above in this proof, for the first cohomology it follows from the

absence of the second cohomology on the curve.) Therefore, using lemma 3, we obtain that the following sequence is exact:

$$0 \to \varprojlim_{j\geq 0} H^0(C, \mathcal{G}_{0,j}) \to \varprojlim_{j\geq 0} H^0(C, \mathcal{O}^*_{X_j}) \to H^0(C, \mathcal{O}^*_C) \to$$
$$\to \varprojlim_{j\geq 0} H^1(C, \mathcal{G}_{0,j}) \to \varprojlim_{j\geq 0} H^1(C, \mathcal{O}^*_{X_j}) \to H^1(C, \mathcal{O}^*_C) \to 0. \tag{13}$$

From exact sequence

$$1 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{A}_0^* \longrightarrow \mathcal{O}_C^* \longrightarrow 1$$

we obtain the following exact sequence:

$$0 \longrightarrow H^0(C, \mathcal{G}_0) \longrightarrow H^0(C, \mathcal{A}_0^*) \longrightarrow H^0(C, \mathcal{O}_C^*) \longrightarrow$$
$$\longrightarrow H^1(C, \mathcal{G}_0) \longrightarrow H^1(C, \mathcal{A}_0^*) \longrightarrow H^1(C, \mathcal{O}_C^*) \longrightarrow 0. \tag{14}$$

We have the natural map of exact sequence (14) to exact sequence (13), and we know that the maps at each term except for one term of the sequence are isomorphisms. But then it follows that the residuary map

$$H^1(C, \mathcal{A}_0^*) \longrightarrow \varprojlim_{j\geq 0} H^1(C, \mathcal{O}^*_{X_j})$$

is also an isomorphism.

$\square$

Corollary 3. *Under conditions of proposition 5 we suppose that C is an affine curve. Then $Pic(X_\infty) = Pic(C)$.*

Proof follows from the proposition and from $H^1(C, \mathcal{G}_{j,j+1}) = 1$ for any $j \geq 0$.

$\square$

Let $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ be a ribbon over the field k. For a point $x \in C$ we denote by $\mathcal{A}_{0,x}$ the local ring which is a stalk of the sheaf $\mathcal{A}_0$ at the point x. Let $\mathcal{M}_x$ be the maximal ideal of $\mathcal{A}_{0,x}$. Further we will need to compare the following two rings.

Definition 8. We denote by $\widehat{\mathcal{A}}_{0,x}$ the $\mathcal{M}_x$-adic completion of the ring $\mathcal{A}_{0,x}$. We denote by $\tilde{\mathcal{A}}_{0,x}$ the ring

$$\tilde{\mathcal{A}}_{0,x} = \varprojlim_i \widehat{\mathcal{O}}_{X_i,x}.$$

Proposition 6. *1. We have the following commutative diagram of morphisms of local rings*

$$\begin{array}{ccc} \mathcal{A}_{0,x} & \to & \widehat{\mathcal{A}}_{0,x} \\ \| & & \downarrow \alpha \\ \mathcal{A}_{0,x} & \to & \tilde{\mathcal{A}}_{0,x} \end{array}$$

where the horizontal arrows are injective.

2. If $\dim_k(\mathcal{M}_x/\mathcal{M}_x^2) < \infty$, then the ring $\tilde{\mathcal{A}}_{0,x}$ is Noetherian and α is surjective, and the Krull dimension of $\tilde{\mathcal{A}}_{0,x}$: $dim\tilde{\mathcal{A}}_{0,x} \geq 2$. Furthermore, $\tilde{I}_j = I_j\tilde{\mathcal{A}}_{0,x}$, where $\tilde{I}_j = Ker(\widetilde{\mathcal{A}}_{0,x} \to \widehat{\mathcal{O}}_{X_i,x})$, $I_j = \mathcal{A}_{j,x}$.

Proof. We prove assertion 1 of the proposition. We define a linear topology on $A := \mathcal{A}_{0,x}$ by taking as open ideals all ideals $\mathcal{Q}$ of finite colength which contain some ideal $I_i := \mathcal{A}_{i,x}$. Thus the set $\{\mathcal{Q}\}$ of ideals contains the ideals $I_i + \mathcal{M}_x^n$ for all i, n, since A/I_i is Noetherian, and so it is coarser or equivalent to the $\mathcal{M}_x$-adic topology, and it is separated (since for any $a \neq 0$ in A there is I_i with $a \neq 0 \mod I_i$, and n with $a \mod I_i \notin \mathcal{M}_x^n(A/I_i)$, hence $a \notin \mathcal{M}_x^n + I_i = \mathcal{Q}$).

Hence assertion 1 follows, since $\tilde{A}$ is the completion of A with respect to the $\{\mathcal{Q}\}$-topology.

We prove assertion 2 of the proposition. We recall that $k(x) = A/\mathcal{M}_x$. If $\dim_{k(x)}(\mathcal{M}_x/\mathcal{M}_x^2) = n < \infty$, then $gr_{\mathcal{M}_x}(A)$ is Noetherian (as an image of a surjection $Sym_{A/\mathcal{M}_x}(\mathcal{M}_x/\mathcal{M}_x^2) \to gr_{\mathcal{M}_x}(A)$) and $\dim_{k(x)}(A/\mathcal{M}_x^{k+1}) \leq C_{n+k}^n$. Therefore $\widehat{A}$ is Noetherian by [2, ch. III, §2.9, corol.2] (since $gr_{\{\widehat{\mathcal{M}_x^i}\}}(\widehat{A}) = gr_{\mathcal{M}}(A)$), and $\widehat{A}$, $\tilde{A}$ both carry a linear topology, which is linearly compact. (The topology of $\widehat{A}$ is linearly compact, since $\dim_k \widehat{A}/\widehat{\mathcal{M}_x^i} < \infty$, see [2, ch.III, §2]). Since α is a continuous homomorphism and A is dence in $\tilde{A}$ and in $\widehat{A}$, it follows that α is surjective with kernel $\cap_{\mathcal{Q}}\widehat{\mathcal{Q}} = \cap_j \widehat{I}_j$ ($\widehat{I}$ is the $\mathcal{M}_x$-adic completion of an ideal). The topology of $\tilde{A}$ is the $\tilde{\mathcal{M}}_x$-adic topology.

Now we prove $\tilde{I}_j = I_j\tilde{A}$. We have $\tilde{I}_j/\tilde{I}_{j+k} = I_j\widehat{A/I_{j+k}} = I_j\tilde{A}/\tilde{I}_{j+k}$, hence $\tilde{I}_j = I_j\tilde{A} + \tilde{I}_{j+k}$ for any $k > 0$. But since $\tilde{A}$ is Noetherian, the ideal $I_j\tilde{A}$ (as any ideal in $\tilde{A}$) is closed in the $\mathcal{M}_x$-adic topology and the $\{\tilde{I}_j\}$-topology is finer (since $\tilde{A}$ is linearly compact). Therefore $\tilde{I}_j = I_j\tilde{A}$.

To prove that $dim\tilde{A} \geq 2$, we choose $u \in \mathcal{M}_x$ which lifts a nonzero divisor of $\mathcal{M}_{C,x} = \mathcal{M}_x/I_1$. We'll prove $l(A/(I_{j+1} + uA)) \geq j+1$. It follows by induction on j from the following exact sequence

$$\begin{array}{ccccccccc} 0 \to & I_j/I_{j+1} & \to & A/I_{j+1} & \to & A/I_j & \to 0 \\ & \downarrow u & & \downarrow u & & \downarrow u & \\ 0 \to & I_j/I_{j+1} & \to & A/I_{j+1} & \to & A/I_j & \to 0 \end{array}$$

since we have statement (3c) of definition 1, so $l((I_j/I_{j+1})/u(I_j/I_{j+1})) \geq 1$, $l((A/I_1)/u(A/I_1)) = l(A/(I_1 + uA)) \geq 1$, $l((A/I_{j+1})/u(A/I_{j+1})) = l((A/I_j)/u(A/I_j)) + l((I_j/I_{j+1})/u(I_j/I_{j+1}))$.

Thus $l(\tilde{A}/(\tilde{I}_j + u\tilde{A})) \geq j$, and $l(\tilde{A}/u\tilde{A}) = \infty$. Since u is not a zero-divisor in $\tilde{A} = \varprojlim \widehat{A/I_j}$, it follows that $dim\tilde{A} > 1$.

$\square$

Corollary 4. *We suppose that* $\dim_{k(x)}(\mathcal{M}_x/\mathcal{M}_x^2) = 2$. *Then we have the following properties under notations of proposition 6.*

1. α *is an isomorphism.*

2. $\tilde{\mathcal{A}}_{0,x}$ is a 2-dimensional regular ring.

Proof. We always have $dim\hat{\mathcal{A}}_{0,x} \ge dim\tilde{\mathcal{A}}_{0,x}$. By [2, ch.III, §3, prop.3], the filtration $\{\widehat{\mathcal{M}_x^i}\}$ is $\widehat{\mathcal{M}_x}$-stable in the ring $\hat{\mathcal{A}}_{0,x}$. Then by [1, prop.11.4, th. 11.14], we have $dim\hat{\mathcal{A}}_{0,x} = \deg \chi_{\mathcal{M}_x}(n) = \deg g(n)$, where $\chi_{\mathcal{M}_x}(n)$, $g(n)$ are characteristic polynomials for the filtrations $\{\widehat{\mathcal{M}_x}^i\}$, $\{\widehat{\mathcal{M}_x^i}\}$, and $2 = dim\, Sym_{A/\mathcal{M}_x}(\mathcal{M}_x/\mathcal{M}_x^2) \ge \deg g(n)$ (since $g(n) \le \chi_\nu(n)$ for all $n \gg 0$, where χ_ν is the characteristic polynomial of the ring $(Sym_{A/\mathcal{M}_x}(\mathcal{M}_x/\mathcal{M}_x^2))_\nu$, where the prime ideal $\nu = \bigoplus_{n=1}^{\infty} S^n(\mathcal{M}_x/\mathcal{M}_x^2)$).

Therefore, using assertion 2 of proposition 6, we have $dim\tilde{\mathcal{A}}_{0,x} = dim\hat{\mathcal{A}}_{0,x} = 2$ and $\tilde{\mathcal{A}}_{0,x}$ is a 2-dimensional regular ring with a prime ideal (0). Therefore, $ker(\alpha)$ must be a prime ideal, hence $ker(\alpha) = 0$, since otherwise $dim\hat{\mathcal{A}}_{0,x} > 2$.

□

Proposition 7. *The group $\mathcal{A}_x^*/\mathcal{A}_{0,x}^*$ is non-trivial if and only if there exists an integer $i > 0$ such that $\mathcal{A}_{i,x}\mathcal{A}_{-i,x} = \mathcal{A}_{0,x}$. In this case the following properties are satisfied.*

1. *All $\mathcal{A}_{j,x}$ ($j \in \mathbb{Z}$) are finitely generated $\mathcal{A}_{0,x}$-modules.*

2. *The invertible sets $\mathcal{A}_{j,x}$'s, i.e. those for which $\mathcal{A}_{j,x}\mathcal{A}_{-j,x} = \mathcal{A}_{0,x}$ form a cyclic group $\{\mathcal{A}_{id,x} | i \in \mathbb{Z}\}$ with some $d > 0$.*

Proof. If $\mathcal{A}_{i,x}\mathcal{A}_{-i,x} = \mathcal{A}_{0,x}$, there are finitely many elements $a_i \in \mathcal{A}_{i,x}$, $b_i \in \mathcal{A}_{-i,x}$, $i = 1, \dots, r$ such that $\sum a_i b_i = 1$.

Since $\mathcal{A}_{0,x}$ is a local ring there is one pair (a_i, b_i) with $a_i b_i \in \mathcal{A}_{0,x}^*$, and so there is a pair (a, b), $a \in \mathcal{A}_{i,x}$, $b \in \mathcal{A}_{-i,x}$ with $ab = 1$.

Now from $ab = 1$, $a \in \mathcal{A}_{i,x}$, $b \in \mathcal{A}_{-i,x}$, we obtain $\mathcal{A}_{i,x} = \mathcal{A}_{0,x}a$, $\mathcal{A}_{-i,x} = \mathcal{A}_{0,x}b$. For, if $a' \in \mathcal{A}_{j,x}$ and $a'b = f \in \mathcal{A}_{0,x}$, then $0 = ((a' - fa)b)a = a' - fa$, hence $a' = fa$. Similarly, $\mathcal{A}_{ki,x} = \mathcal{A}_{0,x}a^k$, $\mathcal{A}_{-ki,x} = \mathcal{A}_{0,x}b^k$, since $a^k b^k = 1$.

If $\mathcal{A}_{i,x}\mathcal{A}_{-i,x} = \mathcal{A}_{0,x}$, $\mathcal{A}_{j,x}\mathcal{A}_{-j,x} = \mathcal{A}_{0,x}$ and $d = gcd(i,j)$, then $\mathcal{A}_{d,x}\mathcal{A}_{-d,x} = \mathcal{A}_{0,x}$. For, if $\mathcal{A}_{i,x} = \mathcal{A}_{0,x}a$, $\mathcal{A}_{j,x} = \mathcal{A}_{0,x}a'$ and $d = mi + nj$, then $a^m a'^n \in \mathcal{A}_{d,x}$, and if $b = a^{-1}$, $b' = a'^{-1}$ $b^m b'^n \in \mathcal{A}_{-d,x}$ and $(a^m a'^n)(b^m b'^n) = 1$.

Thus, assertion 2 of this proposition is proved. To prove assertion 1 of the proposition, we observe that for any $\mathcal{A}_{j,x}$ there is a multiple $k = dj$ of d such that $\mathcal{A}_{k,x} \subset \mathcal{A}_{j,x}$, and $\mathcal{A}_{j,x}/\mathcal{A}_{k,x}$ is a finitely generated $\mathcal{A}_{0,x}$-module.

□

Now we want to discuss the group $H^1(C, \mathcal{A}^*)$.

Proposition 8. *Let $\overset{\circ}{\mathrm{X}}_\infty$ be a ribbon with an irreducible underlying curve C. We assume that the function of order* ord *is a homomorphism from $\mathcal{A}^*(V)$ to $\mathbb{Z}$ for any open $V \subset C$ (see, for example, proposition 9 below). Then we have $\mathcal{A}^*/\mathcal{A}_0^* \subseteq \mathbb{Z}_C$.*

Let the sheaf $\mathcal{A}^*/\mathcal{A}_0^*|_U$ be a constant sheaf for an open set U, which is equal to $d\mathbb{Z}$. (We suppose that this U is maximal.) We have the following.

1. *If $H^0(C, \mathcal{A}^*/\mathcal{A}_0^*) = md\mathbb{Z}$ for some $m \neq 0$, then $H^1(C, \mathcal{A}^*/\mathcal{A}_0^*)$ is a finite abelian group of order less or equal to m^{s-1} if $s > 1$, and is equal to 0 otherwise.*

2. *If $H^0(C, \mathcal{A}^*/\mathcal{A}_0^*) = 0$, then $rk(H^1(C, \mathcal{A}^*/\mathcal{A}_0^*)) \leq s-1$ if $s > 1$, and $H^1(C, \mathcal{A}^*/\mathcal{A}_0^*) = 0$ otherwise.*

In both cases, s is the number of critical points of $\mathcal{A}^*/\mathcal{A}_0^*$, i.e. $s = \sharp(C \setminus U)$.

Proof. If $a \in \mathcal{A}_x^* \cap (\mathcal{A}_{j,x} \backslash \mathcal{A}_{j+1,x})$, where $x \in C$ is a point, and $b \in \mathcal{A}_x^*$ is the inverse of a, then $b \in \mathcal{A}_x^* \cap (\mathcal{A}_{-j,x} \backslash \mathcal{A}_{-j+1,x})$. Then $\mathcal{A}_{j,x} = \mathcal{A}_{0,x} a$. The relations $\mathcal{A}_{j,x} = \mathcal{A}_{0,x} a$, $\mathcal{A}_{-j,x} = \mathcal{A}_{0,x} b$ and $ab = 1$ extend to a neighbourhood U of x.

Since $\mathcal{A}_j/\mathcal{A}_{j+1}$ is a torsion free sheaf, we obtain that if $a \in \mathcal{A}^*(U)$, then there exists a unique $j \in \mathbb{Z}$ such that $a_x \in \mathcal{A}_{j,x} \backslash \mathcal{A}_{j+1,x}$, and the inverse b satisfies $b_x \in \mathcal{A}_{-j,x} \backslash \mathcal{A}_{-j+1,x}$, and $\mathcal{A}_j|_U = \mathcal{A}_0|_U a$, $\mathcal{A}_{-j}|_U = \mathcal{A}_0|_U b$. So we get in this case an injection

$$\mathcal{A}^*/\mathcal{A}_0^* \rightarrow \mathbb{Z}_C, \quad a \mapsto j = \mathrm{ord}(a)$$

($\mathbb{Z}_C$ is a constant sheaf on C). This is an isomorphism iff $\mathcal{A}_1, \mathcal{A}_{-1}$ are mutually dual invertible $\mathcal{A}_0$-modules.

By our assumptions we have that either $\mathcal{A}^* = \mathcal{A}_0^*$, or there is a smallest positive integer d such that there exists a point x and $a \in \mathcal{A}_x^*$ of order d. Then there is a largest open set U where $\mathcal{A}_d, \mathcal{A}_{-d}$ are invertible mutually dual.

Then $\mathcal{A}^*/\mathcal{A}_0^* \subset d\mathbb{Z}$ and the cokernel is a sheaf with support in $C \backslash U$. If $H^0(C, \mathcal{A}^*/\mathcal{A}_0^*) = 0$, then at least one stalk of the sheaf $d\mathbb{Z}/(\mathcal{A}^*/\mathcal{A}_0^*)$ in these points is $d\mathbb{Z}$. If $H^0(C, \mathcal{A}^*/\mathcal{A}_0^*) = md\mathbb{Z}$, the stalks of the sheaf $d\mathbb{Z}/(\mathcal{A}^*/\mathcal{A}_0^*)$ in these points are finite groups, whose order is less or equal to m. Now, using the long cohomology sequence of the short sequence

$$0 \rightarrow \mathcal{A}^*/\mathcal{A}_0^* \xrightarrow{\mu} d\mathbb{Z} \rightarrow coker(\mu) \rightarrow 0$$

we obtain the proof. (We use that the first cohomology of a constant sheaf on an irreducible space is equal to zero in Zariski topology.)

$\square$

Proposition 9. *Let $\overset{\circ}{\mathrm{X}}_\infty$ be a ribbon with an irreducible underlying curve C over a field k. Assume that there exists a point $x \in C$ such that $\mathcal{A}_{1,x}\mathcal{A}_{-1,x} = \mathcal{A}_{0,x}$. Then the function of order* ord *is compatible with the restriction homomorphisms $\mathcal{A}^*(U) \rightarrow \mathcal{A}^*(V)$ for open $V \subset U$, and the function of order* ord_U *is a homomorphism from $\mathcal{A}^*(U)$ to $\mathbb{Z}$ for any open U.*

Proof. As it was shown in the proof of proposition 7, there is an invertible element $a \in \mathcal{A}_{1,x} \backslash \mathcal{A}_{2,x}$ such that $a^{-1} \in \mathcal{A}_{-1,x}$. So, there exists an open $U \ni x$ such that $a \in \mathcal{A}_1(U)$, $a^{-1} \in \mathcal{A}_{-1}(U)$.

Now we need the following lemma.

Lemma 4. *We consider a ribbon $(C, \mathcal{A})$, where C is an irreducible curve over a field k. Let the sheaf $\mathcal{A}$ satisfies the condition (*) (see definition 5) with the following extra*

property: for any open U_α from (*) there exists an invertible section $a \in \mathcal{A}_1(U_\alpha)\backslash\mathcal{A}_2(U_\alpha)$ such that $a^{-1} \in \mathcal{A}_{-1}(U_\alpha)$.

Then the function of order ord *is compatible with the restriction homomorphisms $\mathcal{A}^*(U) \to \mathcal{A}^*(V)$ for open $V \subset U$, and the function of order* ord_U *is a homomorphism from $\mathcal{A}^*(U)$ to $\mathbb{Z}$ for any open U.*

Proof. The first assertion of the lemma follows from the second one. Indeed, if $V \subset U$ are two open subsets and $b \in \mathcal{A}^*(U)$, $\mathrm{ord}_U(b) = k$, then $\mathrm{ord}_U(b^{-1}) = -k$. We always have $\mathrm{ord}_V(b|_V) \geq \mathrm{ord}_U(b)$. If we suppose that $\mathrm{ord}_V(b|_V) > \mathrm{ord}_U(b)$, then $\mathrm{ord}_V((b|_V)^{-1}) < -k = \mathrm{ord}_U(b^{-1})$. But $(b|_V)^{-1} = b^{-1}|_V$ and $\mathrm{ord}_V(b^{-1}|_V) \geq \mathrm{ord}_U(b^{-1}) = -k$, we have a contradiction.

Now we prove the second assertion of the lemma. At first, we prove it for each U_α. We note that for any $b \in \mathcal{A}^*(U_\alpha)$ and any $k \in \mathbb{Z}$ we have $\mathrm{ord}_{U_\alpha}(ba^k) = \mathrm{ord}_{U_\alpha}(b) + k$, where a is an invertible element from $\mathcal{A}_1(U_\alpha)\backslash\mathcal{A}_2(U_\alpha)$ such that $a^{-1} \in \mathcal{A}_{-1}(U_\alpha)$. Indeed, by definition of a ribbon, we always have $\mathrm{ord}_{U_\alpha}(bc) \geq \mathrm{ord}_{U_\alpha}(b) + \mathrm{ord}_{U_\alpha}(c)$ for any $b, c \in \mathcal{A}^*(U_\alpha)$. Let $\mathrm{ord}_{U_\alpha}(ba) > \mathrm{ord}_{U_\alpha}(b) + 1$. Then

$$\mathrm{ord}_{U_\alpha}(b) = \mathrm{ord}_{U_\alpha}(baa^{-1}) \geq \mathrm{ord}_{U_\alpha}(ba) - 1 > \mathrm{ord}_{U_\alpha}(b),$$

we have a contradiction.

We note that $\mathrm{ord}_{U_\alpha}(bc) = \mathrm{ord}_{U_\alpha}(b) + \mathrm{ord}_{U_\alpha}(c)$ if $\mathrm{ord}_{U_\alpha}(c) = 0$. Indeed, if $\mathrm{ord}_{U_\alpha}(bc) > \mathrm{ord}_{U_\alpha}(b)$, then this would mean that $\bar{b}\bar{c} = 0$, where $\bar{b} \in \mathcal{A}_{\mathrm{ord}(b)}(U_\alpha)/\mathcal{A}_{\mathrm{ord}(b)+1}(U_\alpha)$, $\bar{c} \in \mathcal{O}_C(U_\alpha)\mathcal{A}_0(U_\alpha)/\mathcal{A}_1(U_\alpha)$. But $\bar{c}, \bar{b} \neq 0$, and $\mathcal{A}_{\mathrm{ord}(b)}/\mathcal{A}_{\mathrm{ord}(b)+1}$ is a torsion free sheaf by definition, therefore we obtain a contradiction.

For any $b, c \in \mathcal{A}^*(U_\alpha)$ we have

$$\mathrm{ord}_{U_\alpha}(bc) = \mathrm{ord}_{U_\alpha}(ba^{-\mathrm{ord}(b)}a^{\mathrm{ord}(b)}c) = \mathrm{ord}_{U_\alpha}(ba^{-\mathrm{ord}(b)}) + \mathrm{ord}_{U_\alpha}(a^{\mathrm{ord}(b)}c) =$$

$$\mathrm{ord}_{U_\alpha}(b) + \mathrm{ord}_{U_\alpha}(c).$$

The arguments from the beginning of the proof show that for any open $V \subset U_\alpha$ $\mathrm{ord}_V(a|_V) = 1$, and $\mathrm{ord}_V((a|_V)^{-1}) = -1$. Therefore, ord_V is also a homomorphism on $\mathcal{A}^*(V)$.

Now let U be an arbitrary open nonempty subset of C. Then $U = \cup_\alpha(U \cap U_\alpha)$, and $\mathrm{ord}_{U\cap U_\alpha}$ is a homomorphism for each α. Let $b \in \mathcal{A}^*(U)$, $\mathrm{ord}_U(b) = k$. Assume that there exists β such that $\mathrm{ord}_{U\cap U_\beta}(b|_{U\cap U_\beta}) = l > k$. Then for any α we have $U \cap U_\beta \cap U_\alpha \neq \varnothing$ and $\mathrm{ord}_{U\cap U_\beta\cap U_\alpha}(b|_{U\cap U_\beta \cap U_\alpha}) = l$ and therefore $\mathrm{ord}_{U\cap U_\alpha}(b|_{U\cap U_\alpha}) = l$. Since $\mathcal{A}_l$ is a subsheaf of $\mathcal{A}$, this would mean that $b \in \mathcal{A}_l(U)$, we have a contradiction.

So, we have for any $b, c \in \mathcal{A}^*(U)$

$$\mathrm{ord}_U(bc) = \mathrm{ord}_{U\cap U_\alpha}((bc)|_{U\cap U_\alpha})\, \mathrm{ord}_{U\cap U_\alpha}(b|_{U\cap U_\alpha}) + \mathrm{ord}_{U\cap U_\alpha}(c|_{U\cap U_\alpha}) = \mathrm{ord}_U(b) + \mathrm{ord}_U(c).$$

The lemma is proved.

$\square$

By lemma the function of order is a homomorphism on U and on all open subsets of U. Let $V \subset C$, $V \neq C$, $V \nsubseteq U$ be an open set. Since C is a reduced irreducible curve, V must be affine and $V \cap U$ is also affine. Without loss of generality we can assume that $V \cap U = D(f')$, where $V = \underline{\mathrm{Spec}}(B)$, $f' \in B$, $B = \mathcal{O}_C(V)$. Let f be a representative of f' in $\mathcal{A}_0(V)$. Clearly, it is invertible in $\mathcal{A}_0(V \cap U)$. Let $b = a|_{V \cap U}$. We know that b is invertible, $\mathrm{ord}(b) = 1$, $\mathrm{ord}(b^{-1}) = -1$. Since the sheaves $\mathcal{A}_1/\mathcal{A}_2$, $\mathcal{A}_{-1}/\mathcal{A}_0$ are coherent and $V \cap U$ is affine, there exists natural n such that

$$f^n b \mod \mathcal{A}_2(V \cap U) = \phi_{VD(f')}(\bar{b}), \qquad f^n b^{-1} \mod \mathcal{A}_0(V \cap U)\phi_{VD(f')}(\overline{b^{-1}}),$$

where $\phi_{VD(f')} : \mathcal{A}(V) \to \mathcal{A}(D(f'))$ is the restriction homomorphism and $\bar{b} \in \mathcal{A}_1/\mathcal{A}_2(V)$, $\overline{b^{-1}} \in \mathcal{A}_{-1}/\mathcal{A}_0(V)$, by [6, lemma 5.3] and by proposition 3. Let $\tilde{b}, \widetilde{b^{-1}}$ be representatives of $\bar{b}, \overline{b^{-1}}$ in $\mathcal{A}_1(V)$, $\mathcal{A}_{-1}(V)$ correspondingly. Then we have $\mathrm{ord}_V(\tilde{b}\widetilde{b^{-1}}) \geq 0$ and

$$\mathrm{ord}_V(\tilde{b}\widetilde{b^{-1}}) \leq \mathrm{ord}_{V \cap U}(\phi_{VD(f')}(\tilde{b}\widetilde{b^{-1}}))$$

by the properties of $\mathcal{A}$. But $\phi_{VD(f')}(\tilde{b}\widetilde{b^{-1}}) = f^{2n} \mod \mathcal{A}_1(V \cap U)$, wherefrom $\mathrm{ord}_V(\tilde{b}\widetilde{b^{-1}})0$.

Note that for any $d \in \mathcal{A}(V)$ we have $\mathrm{ord}_V(dc) = \mathrm{ord}_V(d) + \mathrm{ord}_V(c)$ if $\mathrm{ord}_V(c) = 0$. Indeed, if $\mathrm{ord}_V(dc) > \mathrm{ord}_V(d)$, then this would mean that $\bar{d}\bar{c} = 0$, where $\bar{d} \in \mathcal{A}_{\mathrm{ord}(d)}(V)/\mathcal{A}_{\mathrm{ord}(d)+1}(V)$, $\bar{c} \in \mathcal{O}_C(V)$. But the curve C is reduced and irreducible and $\mathcal{A}_{\mathrm{ord}(d)}/\mathcal{A}_{\mathrm{ord}(d)+1}$ is a torsion free sheaf by definition, wherefrom we obtain a contradiction.

Now, repeating the arguments from the proof of lemma 4, we obtain $\mathrm{ord}_V(d\tilde{b}^k) = \mathrm{ord}_V(d) + k$ for any integer k and for any $d, c \in \mathcal{A}^*(V)$

$$\mathrm{ord}_V(dc) = \mathrm{ord}_V(d\widetilde{b^{-1}}^{\mathrm{ord}(d)}\tilde{b}^{\mathrm{ord}(d)}c)\,\mathrm{ord}_V(d\widetilde{b^{-1}}^{\mathrm{ord}(d)}) + \mathrm{ord}_V(\tilde{b}^{\mathrm{ord}(d)}c) = \mathrm{ord}_V(d) + \mathrm{ord}_V(c).$$

At last, if $V = C$, then we can apply the arguments at the end of the proof of lemma 4. The proposition is proved.

□

Corollary 5. *If there exists a point P on an irreducible curve C such that $\mathcal{A}_{1,P}\mathcal{A}_{-1,P} = \mathcal{A}_{0,P}$, then the following properties are satisfied.*

1. *The embedding of sheaves $\mathcal{A}^*/\mathcal{A}_0^* \overset{\mathrm{ord}}{\to} \mathbb{Z}_C$ is an isomorphism on an open subset $U \subset C$. Besides, in the remaining points of $C \backslash U$, the stalks $(\mathcal{A}^*/\mathcal{A}_0^*)_x$ are cyclic subgroups $d_x\mathbb{Z}$ of $\mathbb{Z}$. If $H^0(C, \mathcal{A}^*/\mathcal{A}_0^*) = d\mathbb{Z}$ with $d > 0$, then all d_x are divisors of d.*

2. *If P is a smooth point on the curve C, then $\dim_{k(P)}(\mathcal{M}_P/\mathcal{M}_P^2) = 2$.*

Proof. The proof of assertion 1 of this corollorary is clear.

Now we prove assertion 2. From proposition 7 we know that in our case $\mathcal{A}_{i,P} = \mathcal{A}_{0,P}a^i$ for all $i \geq 1$. Since P is a smooth point, we have $\mathcal{M}_{C,P} = \mathcal{O}_{C,P}\bar{u}$ for some $\bar{u} \in \mathcal{M}_{C,P}$. Let $u \in \mathcal{A}_{0,P}$ be a representative of $\bar{u}$. Then, clearly, u, a generate the ideal $\mathcal{M}_P$ in the ring $\mathcal{A}_{0,P}$ and are linearly independent in $\mathcal{M}_P/\mathcal{M}_P^2$. So, we conclude that $\dim_{k(P)}(\mathcal{M}_P/\mathcal{M}_P^2) = 2$.

$\square$

Example 7. If a curve C is not irreducible, then it is possible that the function of order is not a homomorphism from $\mathcal{A}(U)^*$ to $\mathbb{Z}$ for open $U \subset C$.

For example, if we take an algebraizable ribbon from example 1, where X is an affine plane and C is a curve given by the equation $xy = 0$, then the elements x and y will be invertible of order zero elements for any open $U \subset C$ such that U contains the point $(x = 0, y = 0)$. For, (xy) is an invertible element from $\mathcal{A}(U)$, and therefore $x^{-1} = y(xy)^{-1}$, $y^{-1} = x(xy)^{-1}$. But $\mathrm{ord}_U(xy) = 1$, so, ord_U is not a homomorphism.

Example 8. Let $\overset{\circ}{X}_\infty$ be a ribbon from example 1, where X is assumed to be a smooth projective surface. Assume also that $(C \cdot C) \neq 0$, and C is an irreducible curve. We have that the condition $\mathcal{A}_{1,P}\mathcal{A}_{-1,P} = \mathcal{A}_{0,P}$ of corollary 5 is satisfied at each point $P \in C$. Therefore, by proposition 8 and corollary 5 we have the following exact sequence of sheaves on C:

$$1 \longrightarrow \mathcal{A}_0^* \longrightarrow \mathcal{A}^* \longrightarrow \mathbb{Z} \longrightarrow 0,$$

and $H^0(C, \mathcal{A}^*/\mathcal{A}_0^*) = \mathbb{Z}$, $H^1(C, \mathcal{A}^*/\mathcal{A}_0^*) = 0$. It gives the following exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\alpha} Pic(X_\infty) \to Pic(\overset{\circ}{X}_\infty) \to 0,$$

where $\alpha(1) = \mathcal{A}_1$. (The map α is an injective map, because $\alpha(1)$ is not a torsion element in the group $Pic(X_\infty)$. Indeed, the image of $\alpha(1)$ in $Pic(C)$ has degree equal to $-(C \cdot C) \neq 0$.) So, we obtain that $Pic(\overset{\circ}{X}_\infty) \simeq Pic(X_\infty)/\langle \mathcal{A}_1 \rangle \simeq Pic(X_\infty)/\mathbb{Z}$.

For each i we have the exact sequence

$$0 \to H^1(C, \frac{1 + \mathcal{A}_1}{1 + \mathcal{A}_{i+1}}) \to Pic(X_i) \to Pic(C) \to 0$$

and therefore we have the map

$$Pic(X_i) \xrightarrow{deg} \mathbb{Z} \to 0, \quad \mathcal{L} \mapsto deg(\mathcal{L}|_C).$$

By our assumptions we have $deg(\mathcal{A}_1/\mathcal{A}_{i+1}) = d = -(C \cdot C) \neq 0$. Therefore, we have the following exact diagrams for each i:

$$\begin{array}{ccccccc}
0 \to & Pic^0(X_i) & \to & Pic(X_i) & \to & \mathbb{Z} & \to 0 \\
 & \uparrow & & \bigcup & & \bigcup & \\
 & 0 & \to & \langle \mathcal{A}_1/\mathcal{A}_{i+1} \rangle & \simeq & d\mathbb{Z} & \\
 & & & \uparrow & & \uparrow & \\
 & & & 0 & & 0 &
\end{array}$$

whence

$$0 \to Pic^0(X_i) \to Pic(X_i)/\langle \mathcal{A}_1/\mathcal{A}_{i+1} \rangle \to \mathbb{Z}/d\mathbb{Z} \to 0.$$

Projective system $(Pic(X_i), i \geq 0)$ satisfies the ML-condition (as the first cohomology on the curve), therefore $(Pic^0(X_i), i \geq 0)$ satisfies the ML-condition (as the kernels of the maps to $\mathbb{Z}$). Passing to the projective limit we obtain the exact sequence

$$\begin{array}{ccccccc}
0 \to & Pic^0(X_\infty) & \to & Pic(X_\infty)/\langle \mathcal{A}_1 \rangle & \to & \mathbb{Z}/d\mathbb{Z} & \to 0 \\
 & & & \| & & & \\
 & & & Pic(\overset{\circ}{X}_\infty) & & &
\end{array}$$

In particular, when $X = \mathbb{P}^2$, $C = \mathbb{P}^1 \subset X$, we have $d = -1$ and therefore $Pic^0(X_\infty) \simeq Pic(\mathring{X}_\infty)$. On $Pic^0(X_\infty)$ there exists a structure of a scheme (see, for example, [9]), so in this case there is a structure of a scheme also on $Pic(\mathring{X}_\infty)$.

5 A generalized Krichever-Parshin map

Let $\mathring{X}_\infty = (C, \mathcal{A})$ be a ribbon over a field k.

Definition 9. We say that a point $P \in C$ is a smooth point of the ribbon $\mathring{X}_\infty$ if the following conditions are satisfied.

1. P is a smooth point of C.
2. $(\widehat{\mathcal{A}_i/\mathcal{A}_{i+1}})_P \otimes (\widehat{\mathcal{A}_j/\mathcal{A}_{j+1}})_P \to (\widehat{\mathcal{A}_{i+j}/\mathcal{A}_{i+j+1}})_P$ is an isomorphism of $\widehat{\mathcal{O}}_{C,P}$-modules, and this map is induced by the map from the definition of ribbon: $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$.

Example 9. Let $\mathring{X}_\infty$ be a ribbon from example 1, where $P \in C$ is a smooth point on the curve C and the surface X. Then it is clear that P is a smooth point of the ribbon $\mathring{X}_\infty$.

Remark 7. Immediately from definition follows that a ribbon with a smooth point, whose topological space is irreducible, satisfies the conditions of proposition 9. On the other hand, all the ribbons from examples 2 and 3 have open neighborhoods, where they have smooth points and are algebraizable.

Proposition 10. *Let P be a smooth k-point of the ribbon $(C, \mathcal{A})$. Then*

$$\tilde{\mathcal{A}}_{0,P} \simeq \hat{\mathcal{A}}_{0,P} \simeq k[[u,t]],$$

where $t\tilde{\mathcal{A}}_{0,P} = \tilde{\mathcal{A}}_{1,P}$ and $\widehat{\mathcal{O}}_{C,P} \simeq k[[\tau(u)]]$, where $\tau : \tilde{\mathcal{A}}_{0,P} \to \widehat{\mathcal{O}}_{C,P}$ is a canonical map.

Proof. The isomorphism $\tilde{\mathcal{A}}_{0,P} \simeq \hat{\mathcal{A}}_{0,P}$ follows from corollaries 5 and 4. Now we will prove that $\widehat{\mathcal{O}}_{X_i,P} \simeq k[[u]][t]/t^i$ for some u, t. The proof is by induction on i.

If $i = 1$, then $\widehat{\mathcal{O}}_{X_1,P} = \widehat{\mathcal{O}}_{C,P} \simeq k[[u]]$ for some u. Suppose we have proved the assertion for $(i-1)$. We have the exact triple:

$$0 \to (\widehat{\mathcal{A}_{i-1}/\mathcal{A}_i})_P \to \widehat{\mathcal{O}}_{X_i,P} \xrightarrow{\gamma} \widehat{\mathcal{O}}_{X_{i-1},P} \to 0.$$

Let $\tilde{u}, \tilde{t} \in \widehat{\mathcal{O}}_{X_i,P}$ be elements with $\gamma(\tilde{u}) = u$, $\gamma(\tilde{t}) = t$. From the definition of a smooth point it follows that $\tilde{t}^{i-1}$ is a generator of the $\widehat{\mathcal{O}}_{C,P}$-module $(\widehat{\mathcal{A}_{i-1}/\mathcal{A}_i})_P$. Therefore, $\widehat{\mathcal{O}}_{X_i,P} \simeq k[[u]][t]/t^i$.

Passing to the projective limit with respect to i we are done.

□

Definition 10. Any elements u, t from proposition 10 are called *formal local parameters* of the ribbon $(C, \mathcal{A})$ at the smooth point P.

Lemma 5. *Let $(C, \mathcal{A}, P, u, t)$ be a ribbon over a field k with a smooth k-point P and formal local parameters u, t. Then $u \in \tilde{\mathcal{A}}_{0,P}$ defines an effective Cartier divisor $p_{u,i}$ on the scheme X_i for any i such that $\theta_i^* p_{u,i} = p_{u,i-1}$ and $p_{u,1} = P$, where*

$$\theta_i : X_{i-1} \to X_i$$

is a canonical map.

Proof. We know by proposition 10 that $\widehat{\mathcal{O}}_{X_i,P} \simeq k[[u]][t]/t^i$, because P is a smooth point of the ribbon $(C, \mathcal{A})$. By $\tilde{p}_{u,i} = u \cdot k[[u]][t]/t^i$ we denote the ideal in $\widehat{\mathcal{O}}_{X_i,P}$. Let $p'_{u,i} := \tilde{p}_{u,i} \cap \mathcal{O}_{X_i,P}$ be the ideal in $\mathcal{O}_{X_i,P}$.

We have for some $j > 0$ that $\mathcal{M}_P^j \cdot k[[u]][t]/t^i \subset \tilde{p}_{u,i}$, where $\mathcal{M}_P$ is the maximal ideal of $\mathcal{O}_{X_i,P}$. Therefore, $\mathcal{M}_P^j \mathcal{O}_{X_i,P} \subset p'_{u,i}$. Let $\tilde{u} \in \mathcal{O}_{X_i,P}$ be an element such that $\beta(\tilde{u})$ coincides with $\hat{\beta}(u)$, where $\beta, \hat{\beta}$ are the following natural maps

$$\begin{array}{ccccc}
\beta & : & \mathcal{O}_{X_i,P} & \longrightarrow & \mathcal{O}_{X_i,P}/\mathcal{M}_P^j \\
 & & & & \| \\
\hat{\beta} & : & \widehat{\mathcal{O}}_{X_i,P} & \longrightarrow & \widehat{\mathcal{O}}_{X_i,P}/\widehat{\mathcal{M}}_P^j
\end{array}$$

Then $\tilde{u} \in p'_{u,i}$ and $\tilde{u} \cdot \widehat{\mathcal{O}}_{X_i,P} = \tilde{p}_{u,i}$. Therefore, $p'_{u,i} \cdot \widehat{\mathcal{O}}_{X_i,P} = \tilde{p}_{u,i}$, and $p'_{u,i} = \tilde{u}\mathcal{O}_{X_i,P}$ defines the effective Cartier divisor $p_{u,i}$ in some affine open neigbourhood of the point $P \in X_i$ (and on X_i). By construction, $\theta_i^* p_{u,i} = p_{u,i-1}$.

$\square$

Remark 8. By construction of the ideal $p'_{u,i}$ (or divisor $p_{u,i}$) we obtain that it is uniquely defined by the properties $p'_{u,i} \cdot \widehat{\mathcal{O}}_{X_i,P} = u \cdot \widehat{\mathcal{O}}_{X_i,P}$, $\theta_i^* p_{u,i} = p_{u,i-1}$, and $p'_{u,1} = \mathcal{M}_P \subset \mathcal{O}_{C,P}$.

Definition 11. Let $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ be a ribbon over a scheme S. We say that $\mathcal{N}$ is a torsion free sheaf of rank r on $\overset{\circ}{X}_\infty$ if $\mathcal{N}$ is a sheaf of $\mathcal{A}$-modules on C with a descending filtration $(\mathcal{N}_i)_{i \in \mathbb{Z}}$ of $\mathcal{N}$ by $\mathcal{A}_0$-submodules which satisfies the following axioms.

1. $\mathcal{N}_i \mathcal{A}_j \subseteq \mathcal{N}_{i+j}$ for any i, j.
2. For each i the sheaf $\mathcal{N}_i/\mathcal{N}_{i+1}$ is a coherent sheaf on C, flat over S, and for any $s \in S$ the sheaf $\mathcal{N}_i/\mathcal{N}_{i+1}|_{C_S}$ has no coherent subsheaf with finite support, and is isomorphic to $\mathcal{O}_{C_S}^{\oplus r}$ on a dense open set.
3. $\mathcal{N} = \varinjlim_i \mathcal{N}_i$ and $\mathcal{N}_i = \varprojlim_{j>0} \mathcal{N}_i/\mathcal{N}_{i+j}$ for each i.

Remark 9. It follows from assertion 2 of definition 11 that if C_s (for $s \in S$) is an irreducible curve, then the sheaf $\mathcal{N}_i/\mathcal{N}_{i+1} \mid_{C_s}$ is a rank r torsion free sheaf on C_s for any $i \in \mathbb{Z}$.

Remark 10. If the sheaf $\mathcal{A}$ of a ribbon $\overset{\circ}{X}_\infty$ satisfies the condition (*) from definition 5, then any torsion free sheaf $\mathcal{N}$ of rank r on $\overset{\circ}{X}_\infty$ is coherent. The proof of this fact is the same as in lemma 2.

On the other hand, if $\mathcal{A}$ is only coherent, then there exists a torsion free sheaf that is not coherent, as it follows from the example below.

Example 10. Consider the ribbon $(C, \mathcal{A} = \mathcal{O}_C((t))^Q)$ from example 3. The sheaf $\mathcal{A}$ is coherent, but not weakly Noetherian. The sheaf $\mathcal{N} := \mathcal{O}_C((t))$ with obvious filtration is a torsion free sheaf of rank 1 on $\overset{\circ}{X}_\infty$. But the stalk $\mathcal{N}_Q$ can not be finitely generated: for any finite number of sections $g_1, \dots, g_k \in \mathcal{N}(V)$, $Q \in V$ there are infinite number of elements t^l, $l \ll 0$ that can not be generated by g_i. So, $\mathcal{N}$ is not of finite type and therefore is not coherent.

Definition 12. Let $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ be a ribbon over a field k. We say that a point $P \in C$ is a smooth point of a torsion free sheaf $\mathcal{N}$ on $\overset{\circ}{X}_\infty$ if the following conditions are satisfied.

1. P is a smooth point of $\overset{\circ}{X}_\infty$.
2. $(\widehat{\mathcal{N}_i/\mathcal{N}_{i+1}})_P \otimes_{\hat{\mathcal{O}}_{C,P}} (\widehat{\mathcal{A}_j/\mathcal{A}_{j+1}})_P \to (\widehat{\mathcal{N}_{i+j}/\mathcal{N}_{i+j+1}})_P$ is an isomorphism of $\widehat{\mathcal{O}}_{C,P}$-modules, and this map is induced by the map from the definition of $\mathcal{N}$: $\mathcal{N}_i \cdot \mathcal{A}_j \subset \mathcal{N}_{i+j}$.

Similarly to the proposition 10 we have the following proposition.

Proposition 11. *Let P be a smooth point of a torsion free sheaf $\mathcal{N}$ of rank r on a ribbon $\overset{\circ}{\mathrm{X}}_\infty$ over a field k. Then*

$$\widetilde{\mathcal{N}}_{0,P} \simeq \widetilde{\mathcal{A}}_{0,P}^{\oplus r},$$

where $\widetilde{\mathcal{N}}_{0,P} = \varprojlim\limits_{j \ge 0} (\widehat{\mathcal{N}_0/\mathcal{N}_j})_P$.

Proof. By induction on j and using the exact sequence

$$0 \to (\widehat{\mathcal{N}_{j-1}/\mathcal{N}_j})_P \to (\widehat{\mathcal{N}_0/\mathcal{N}_j})_P \to (\widehat{\mathcal{N}_0/\mathcal{N}_{j-1}})_P \to 0$$

we prove that $(\widehat{\mathcal{N}_0/\mathcal{N}_j})_P \simeq (\widehat{\mathcal{A}_0/\mathcal{A}_j})_P^{\oplus r}$. Then we pass to the projective limit.

□

Example 11. Let $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ be a ribbon such that C is irreducible. We suppose that the function of order is a homomorphism (see, for example, proposition 9 above.) Then every element $\mathcal{N} \in Pic(\overset{\circ}{X}_\infty)$ gives a torsion free sheaf of rank 1 on $\overset{\circ}{X}_\infty$ after the fixing of a filtration on $\mathcal{N}$. (All the possible filtrations form a $\mathbb{Z}$-torsor.) Indeed, the function of order ord gives a homomorphism:

$$\gamma \quad : \quad H^1(C, \mathcal{A}^*) \longrightarrow H^1(C, \mathbb{Z}).$$

And the obstruction to find a filtration on $\mathcal{N}$ is $\gamma(\mathcal{N}) \neq 0$. But any local $\mathbb{Z}$-system on irreducible space C is trivial in Zariski topology, i.e. $H^1(C, \mathbb{Z}) = 0$. Thus, we have a filtration on $\mathcal{N}$.

Example 12. Let $\overset{\circ}{X}_\infty$ be a ribbon from example 1. Let E be a locally free sheaf of rank r on the surface X. Then

$$\overset{\circ}{E}_C := \varinjlim_i \varprojlim_j E(iC)/E(jC)$$

is a torsion free sheaf of rank r on $\overset{\circ}{X}_\infty$. Any point $P \in C \subset X$ that is smooth on C and on X will be a smooth point on $\overset{\circ}{E}_C$.

Remark 11. Similarly to definition 8 we have two $\mathcal{A}_{0,P}$-modules: $\widetilde{\mathcal{N}}_{0,P}$ and $\widehat{\mathcal{N}}_{0,P}$, where the latter is the $\mathcal{M}_P$-adic completion of the module $\mathcal{N}_{0,P}$. Using similar arguments as in the the proof of proposition 6, we obtain that if $\dim_k \mathcal{N}_{0,P}/\mathcal{M}_P\mathcal{N}_{0,P} < \infty$, then the natural homomorphism of $\mathcal{A}_{0,P}$-modules

$$\widehat{\mathcal{N}}_{0,P} \xrightarrow{\alpha} \widetilde{\mathcal{N}}_{0,P}$$

is surjective.

If P is a smooth point of the torsion free sheaf $\mathcal{N}$ of rank r, then $\widehat{\mathcal{A}}_{0,P} \simeq \widetilde{\mathcal{A}}_{0,P}$, $\dim_{k(P)} \mathcal{N}_{0,P}/\mathcal{M}_P\mathcal{N}_{0,P} = r$ and therefore α is an isomorphism of $\widetilde{\mathcal{A}}_{0,P}$-modules.

Further we will work with geometric data consisting of a ribbon, a torsion free sheaf on it, formal local parameters at a point P on the ribbon and a formal trivialization of the sheaf at P.

Definition 13. Let $(\overset{\circ}{X}_\infty, \mathcal{N})$, $(\overset{\circ}{X}'_\infty, \mathcal{N}')$ be two ribbons over a field k with two torsion free sheaves of rank r on them. We say that $(\overset{\circ}{X}_\infty, \mathcal{N})$ is isomorphic to $(\overset{\circ}{X}'_\infty, \mathcal{N}')$ if there is an isomorphism

$$\varphi \ : \ \overset{\circ}{X}_\infty \longrightarrow \overset{\circ}{X}'_\infty$$

of ribbons (see definition 2) and an isomorphism

$$\psi \ : \ \mathcal{N}' \to \varphi_*(\mathcal{N})$$

of graded $\mathcal{A}'$-modules, i.e. $\psi(\mathcal{N}'_i) = \varphi_*(\mathcal{N}_i)$ and $\psi(ln) = \varphi^\sharp(l)\psi(n)$ for any sections $n \in \mathcal{N}'(U)$, $l \in \mathcal{A}'(U)$ over open $U \subset C'$.

Definition 14. We will consider the following geometric data $(C, \mathcal{A}, \mathcal{N}, P, u, t, e_P)$, where

- $(C, \mathcal{A})$ is a ribbon over a field k,
- $\mathcal{N}$ is a torsion free sheaf of rank r on $(C, \mathcal{A})$,
- $P \in C$ is a smooth k-point of the sheaf $\mathcal{N}$,
- u, t are formal local parameters of the ribbon at P,
- $e_P : \widetilde{\mathcal{N}}_{0,P} \to \widetilde{\mathcal{A}}_{0,P}^{\oplus r} \simeq k[[u,t]]^{\oplus r}$ is an isomorphism of $\widetilde{\mathcal{A}}_{0,P}$-modules.

We say that $(C,\mathcal{A},\mathcal{N},P,u,t,e_P)$ is isomorphic to $(C',\mathcal{A}',\mathcal{N}',P',u',t',e'_P)$ if there is an isomorphism (see definition 13)

$$(\varphi,\psi)\ :\ (C,\mathcal{A},\mathcal{N})\longrightarrow(C',\mathcal{A}',\mathcal{N}')$$

such that $\varphi(P)=P'$, $\varphi^\sharp_P(t')=t$, $\varphi^\sharp_P(u')=u$, where $\varphi^\sharp_P:\widetilde{\mathcal{A}'}_{0,P}\to\widetilde{\mathcal{A}}_{0,P}$ is an isomorphism of local rings induced by $\varphi^\sharp$, and the diagram

$$\begin{array}{ccc}\widetilde{\mathcal{N}'}_{0,P} & \stackrel{\psi_P}{\to} & \widetilde{\mathcal{N}}_{0,P}\\ \downarrow e'_P & & \downarrow e_P\\ \widetilde{\mathcal{A}'}^{\oplus r}_{0,P} & \stackrel{\varphi^\sharp_P}{\to} & \widetilde{\mathcal{A}}^{\oplus r}_{0,P}\end{array}$$

is commutative, where the isomorphism ψ_P is induced by ψ.

Definition 15. Let $K=k((u))((t))$ be a two-dimensional local field. We define the following k-subspaces of K:

$$\mathcal{O}(n)=t^nk((u))[[t]]$$

for any n. For any k-subspace $W\subset K^{\oplus r}$ and any $j>i\in\mathbb{Z}$ we define

$$W(i,j)=\frac{W\cap\mathcal{O}(i)^{\oplus r}}{W\cap\mathcal{O}(j)^{\oplus r}}.$$

We have the natural isomorphism $\mathcal{O}(i)/\mathcal{O}(j)=k((u))^{\oplus(j-i)}$, therefore $W(i,j)$ is a k-subspace of the space $k((u))^{\oplus r(j-i)}$. We note that the last space has natural locally linearly compact topology.

Definition 16. Let W be a k-subspace of $K^{\oplus r}=k((u))((t))^{\oplus r}$, let A be a k-subalgebra of $K=k((u))((t))$. (We can consider $K^{\oplus r}$ as a K-module, so the product $A\cdot W\subset K^{\oplus r}$ is defined.)

We suppose that $A\cdot W\subset W$, and $A(i,i+1)\subset k((u))$ is a discrete subspace with quotient being linearly compact space, $W(i,i+1)\subset k((u))^{\oplus r}$ is a discrete subspace with quotient being linearly compact for any $i\in\mathbb{Z}$. Then we call the pair of k-subsapces $(A,W)\subset K\oplus K^{\oplus r}$ as a *Schur pair.*

Remark 12. By induction on $j-i>0$ we have that if $W(i,i+1)$, for any $i\in\mathbb{Z}$, are subspaces from definition 16, then $W(i,j)$ is a discrete subspace in $k((u))^{\oplus r(j-i)}$ with quotient being linearly compact for any $j>i$. Similarly, $A(i,j)$ is a discrete subspace in $k((u))^{\oplus(j-i)}$ with quotient being linearly compact for any $j>i$.

Remark 13. Clearly, the subspaces $W(i,i+1)$ (resp. $A(i,i+1)$) from definition 16 satisfy the Fredholm condition with respect to $k[[u]]^{\oplus r}$ (resp. to $k[[u]]$, see introduction), as the analogous subspaces in the construction of the Krichever map in [15], [12], see also [17].

Theorem 1. *The Schur pairs (A,W) from definition 16 are in one-to-one correspondence with data $(C,\mathcal{A},\mathcal{N},P,u,t,e_P)$ from definition 14 up to an isomorphism, where we additionally assume that C is a projective irreducible curve.*

Corollary 6. *k-subalgebras A from definition 16 are in one-to-one correspondence with data $(C, \mathcal{A}, P, u, t)$ up to an isomorphism, where C has to be a projective irreducible curve.*

Proof. The corollary follows from the theorem, if we take $\mathcal{N} = \mathcal{A}$, $W = A$, and $e_P = 1$.

Now we'll prove the theorem. We have the following diagram of maps for any coherent sheaf M on the scheme X_i and for any $i \ge 0$.

$$\begin{array}{ccccc} \Gamma(X_i \backslash P, M) & \stackrel{\alpha_M}{\to} & \Gamma(\mathrm{Spec}\, \mathcal{O}_{X_i,P} \backslash P, M) & \stackrel{\beta_M}{\to} & \Gamma(\mathrm{Spec}\, \widehat{\mathcal{O}}_{X_i,P} \backslash P, M) \\ & & \| & & \| \\ & & M \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i,\eta_i} & & M \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i,\eta_i} \otimes_{\mathcal{O}_{X_i}} \widehat{\mathcal{O}}_{X_i,P} \end{array}$$

where η_i is the generic point of the scheme X_i.

Now let $M = \mathcal{N}_k / \mathcal{N}_{k+i+1}$ for some k. Then, by statement 1 of proposition 3 and statement 2 of proposition 1, M is coherent sheaf on the scheme X_i. By induction on i, we show that the map α_M is an embedding. It is true for $i = 0$, because $\mathcal{N}_k / \mathcal{N}_{k+1}$ is a torsion free sheaf on C. We have the following commutative diagram for arbitrary $i \ge 1$

$$\begin{array}{ccccccc} 0 \to & \Gamma(X_i \backslash P, \mathcal{N}_{i+k}/\mathcal{N}_{i+k+1}) & \to & \Gamma(X_i \backslash P, \mathcal{N}_k/\mathcal{N}_{i+k+1}) & \to & \Gamma(X_i \backslash P, \mathcal{N}_k/\mathcal{N}_{k+i}) & \to 0 \\ & \downarrow \alpha_{\mathcal{N}_{i+k}/\mathcal{N}_{i+k+1}} & & \downarrow \alpha_{\mathcal{N}_k/\mathcal{N}_{i+k+1}} & & \downarrow \alpha_{\mathcal{N}_k/\mathcal{N}_{i+k}} & \\ 0 \to & (\mathcal{N}_{i+k}/\mathcal{N}_{i+k+1})_{\eta_i} & \to & (\mathcal{N}_k/\mathcal{N}_{i+k+1})_{\eta_i} & \to & (\mathcal{N}_k/\mathcal{N}_{k+i})_{\eta_i} & \to 0, \\ & & & & & \| & \\ & & & & & (\mathcal{N}_k/\mathcal{N}_{k+i})_{\eta_{i-1}} & \end{array}$$

since $\mathcal{N}_{i+k}/\mathcal{N}_{i+k+1}$ is a coherent $\mathcal{O}_{X_i}$-module and the $\mathcal{O}_{X_i}$-module structure on $\mathcal{N}_k/\mathcal{N}_{k+i}$ is the same as the $\mathcal{O}_{X_{i-1}}$-module structure. Therefore, by induction hypothesis, the left and right vertical arrows are embeddings. Hence, the middle arrow is also an embedding.

The map $\beta_{\mathcal{N}_k/\mathcal{N}_{i+k+1}}$ is an embedding for the sheaf $\mathcal{N}_k/\mathcal{N}_{i+k+1}$. Therefore, the map $\beta_{\mathcal{N}_k/\mathcal{N}_{i+k+1}} \circ \alpha_{\mathcal{N}_k/\mathcal{N}_{i+k+1}}$ is an embedding for the sheaf $\mathcal{N}_k/\mathcal{N}_{i+k+1}$.

Now we have for $k = 0$

$$\mathcal{A}_0/\mathcal{A}_{i+1} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i,\eta_i} \otimes_{\mathcal{O}_{X_i}} \widehat{\mathcal{O}}_{X_i,P} \simeq k((u))[t]/t^{i+1},$$

because we have fixed the formal local parameters u, t of our ribbon at P.

We have for $k > 0$

$$\mathcal{A}_k/\mathcal{A}_{k+i+1} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i,\eta_i} \otimes_{\mathcal{O}_{X_i}} \widehat{\mathcal{O}}_{X_i,P} \simeq t^k \cdot k((u))[t]/t^{i+1}$$

as an ideal in $\mathcal{A}_0/\mathcal{A}_{i+1+k} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i,\eta_i} \otimes_{\mathcal{O}_{X_i}} \widehat{\mathcal{O}}_{X_i,P} \simeq k((u))[t]/t^{i+k+1}$.

By definition of a smooth point on a ribbon, we have the natural pairing for $k < 0$

$$(\widehat{\mathcal{A}_k/\mathcal{A}_{k+i+1}})_P \otimes (\widehat{\mathcal{A}_{-k}/\mathcal{A}_{-k+i+1}})_P \to (\widehat{\mathcal{A}_0/\mathcal{A}_{i+1}})_P,$$

and the element $t^{-k} \mod \widetilde{\mathcal{A}}_{-k+i,P} \in (\widehat{\mathcal{A}_{-k}/\mathcal{A}_{-k+i+1}})_P$. Then, by induction on $i \ge 0$, we obtain that

$$(\widehat{\mathcal{A}_k/\mathcal{A}_{k+i+1}})_P \xrightarrow{\times t^{-k}} (\widehat{\mathcal{A}_0/\mathcal{A}_{i+1}})_P \simeq k((u))[t]/t^{i+1}$$

is an isomorphism. Therefore, we have

$$\varinjlim_k \varprojlim_{i\geq 0} (\widehat{\mathcal{A}_k/\mathcal{A}_{k+i+1}})_P \simeq k((u))((t)).$$

Besides, $\mathcal{A} = \varinjlim_k \varprojlim_{i\geq 0} \mathcal{A}_k/\mathcal{A}_{k+i+1}$. Therefore, the ring

$$A := \varinjlim_k \varprojlim_{i\geq 0} \Gamma(X_i \backslash P, \mathcal{A}_k/\mathcal{A}_{k+i+1}) \subset k((u))((t))$$

is a k-subalgebra that satisfies the conditions of the theorem.

Analogously, using the trivialization e_P and formal local parameters u, t, we obtain the isomorphism

$$\varinjlim_k \varprojlim_{i\geq 0} (\widehat{\mathcal{N}_k/\mathcal{N}_{k+i+1}})_P \simeq k((u))((t))^{\oplus r}$$

and the subspace

$$W := \varinjlim_k \varprojlim_{i\geq 0} \Gamma(X_i \backslash P, \mathcal{N}_k/\mathcal{N}_{k+i+1}) \subset k((u))((t))^{\oplus r}$$

is a k-subspace that satisfies the conditions of the theorem.

Thus, starting from the geometric data $(C, \mathcal{A}, \mathcal{N}, P, u, t, e_P)$ from definition 14, we have constructed a Schur pair (A, W) from definition 16.

Now we are going to construct a geometric data starting from a Schur pair. At first, we note that

$$\Gamma(\operatorname{Spec} \mathcal{O}_{X_i,P} \backslash P, \mathcal{A}_k/\mathcal{A}_{k+i+1}) = \varinjlim_{n\geq 0} \Gamma(X_i, \mathcal{A}_k/\mathcal{A}_{k+i+1}(np_{u,i})),$$

where $p_{u,i}$ is the effective Cartier divisor on X_i which was constructed in lemma 5 above.

We consider the k-subspaces for $j > i \in \mathbb{Z}$

$$A(i,j) \subset k((u))^{\oplus(j-i)} \quad \text{and}$$

$$U_n(i,j) = u^{-n} \cdot k[[u]]^{\oplus(j-i)} \subset k((u))^{\oplus(j-i)}.$$

If $i = 0$, the space $\bigoplus_{n\geq 0}(U_n(0,j) \cap A(0,j))$ is a graded ring. We put

$$X_{j-1} = \operatorname{Proj}(\bigoplus_{n\geq 0}(U_n(0,j) \cap A(0,j))).$$

The image of the embedding of $\bigoplus_{n\geq 0}(U_{n-1}(0,1) \cap A(0,1))$ in $\bigoplus_{n\geq 0}(U_n(0,1) \cap A(0,1))$ is a homogeneous ideal that determines a point $P \in X_0$.

If $j > i \in \mathbb{Z}$, then $\bigoplus_{n\geq 0}(U_n(i,j) \cap A(i,j))$ is a graded module over the graded ring $\bigoplus_{n\geq 0}(U_n(0,j-i) \cap A(0,j-i))$. Then we define

$$\mathcal{A}(i,j) = \widetilde{\bigoplus_{n\geq 0}(U_n(i,j) \cap A(i,j))},$$

i.e. it is a coherent sheaf on $X_{(j-i)}$ which is associated with the corresponding graded module. Since there is no zero divisors in the field $k((u))$, the sheaf $\mathcal{A}(i,i+1)$ is a torsion free sheaf on C for any i.

For all $j > i \in \mathbb{Z}$ we have surjective morphisms $\mathcal{A}(i,j+1) \to \mathcal{A}(i,j)$ and injective morphisms $\mathcal{A}(i,j) \to \mathcal{A}(i-1,j)$. Also, from definitions, we have maps for all $i < j$, $k < l$

$$A(i,j) \otimes_k A(k,l) \longrightarrow A(i+k, \min(j+k, i+l)), \tag{15}$$

which are also well-defined maps, if we pass to projective limits with respect to j and l.

So, we define sheaves $\mathcal{A}$, $\mathcal{A}_i$, $i \in \mathbb{Z}$ by

$$\mathcal{A} := \varinjlim_{i} \varprojlim_{j \geq i} \mathcal{A}(i,j), \quad \mathcal{A}_i = \varprojlim_{j \geq i} \mathcal{A}(i,j).$$

The map given by formula (15) defines the multiplication $\mathcal{A}_i \cdot \mathcal{A}_j \subset \mathcal{A}_{i+j}$. Besides, $\widetilde{\mathcal{A}}_{0,P} = k[[u,t]]$, and therefore u,t are the formal local parameters of the ribbon $(X_0, \mathcal{A})$ at the point P.

Analogously, we define sheaves of modules $\mathcal{N}$, $\mathcal{N}_i$, $i \in \mathbb{Z}$ by

$$\mathcal{N} := \varinjlim_{i} \varprojlim_{j \geq i} \mathcal{N}(i,j), \qquad \mathcal{N}_i = \varprojlim_{j \geq i} \mathcal{N}(i,j),$$

where $\mathcal{N}(i,j) = \widetilde{N}$, $N = \bigoplus_{n \geq 0}((u^{-n} \cdot k[[u]]^{\oplus r(j-i)}) \cap W(i,j))$, i.e. $\mathcal{N}(i,j)$ is a coherent sheaf of $\mathcal{O}_{X_{(j-i-1)}}$-modules, which is associated with the corresponding graded module, for all $j > i$. By construction, we have a natural isomorphism

$$e_P \ : \ \widetilde{\mathcal{N}}_{0,P} \to k[[u,t]]^{\oplus r}.$$

The map $(A, W) \mapsto (X_0, \mathcal{A}, \mathcal{N}, P, u, t, e_P)$ just constructed is the inverse to the map which was constructed in the first part of proof of this theorem, because the sheaf $\mathcal{N}_i/\mathcal{N}_j \simeq \Gamma_*(\widetilde{\mathcal{N}_i/\mathcal{N}_{j+1}})$ for all $j > i \in \mathbb{Z}$, where the graded module

$$\Gamma_*(\mathcal{N}_i/\mathcal{N}_j) = \bigoplus_{n \geq 0} \Gamma(X_{j-i-1}, \mathcal{N}_i/\mathcal{N}_j(np_{u,j-i-1})),$$

defines a coherent sheaf on the scheme

$$X_{j-i-1} = \mathrm{Proj}(\bigoplus_{n \geq 0} \Gamma(X_{j-i-1}, \mathcal{O}_{X_{j-i-1}}(np_{u,j-i-1}))),$$

since $\mathcal{O}_{X_{j-i-1}}(p_{u,j-i-1})$ is an ample sheaf on X_{j-i-1}. The latter follows from the lemma below.

Lemma 6. *For any $i > 0$ the sheaf $\mathcal{O}_{X_i}(p_{u,i})$ is an ample sheaf on X_i.*

Proof. X_i is a proper scheme (as X_0 is a projective curve). So, by [6, ch.III, prop.5.3] it is enough to prove that for any $l > 0$, for any coherent sheaf $\mathcal{F}$ on X_i there exists $n_0 > 0$ such that for any $n > n_0$ $H^l(X_i, \mathcal{F} \otimes \mathcal{O}_{X_i}(np_{u,i})) = 0$.

We use induction on i. If $i = 1$, then $p_{u,1}$ is the point P on the projective curve $C = X_0$, i.e. it is an ample divisor. If $i > 1$, we consider the exact sequence of $\mathcal{O}_{X_i}$-modules

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_{X_i}} \mathcal{A}_{i-1}/\mathcal{A}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i}(np_{u,i}) \to \mathcal{F} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i}(np_{u,i}) \to \mathcal{F} \otimes_{\mathcal{O}_{X_i}} (\mathcal{A}_0/\mathcal{A}_{i-1}) \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i}(np_{u,i}) \to 0.$$

The $\mathcal{O}_{X_i}$-module structure of modules $\mathcal{F} \otimes_{\mathcal{O}_{X_i}} \mathcal{A}_{i-1}/\mathcal{A}_i$ and $\mathcal{F} \otimes_{\mathcal{O}_{X_i}} (\mathcal{A}_0/\mathcal{A}_{i-1})$ coincides with the $\mathcal{O}_{X_{i-1}}$-module structure. Therefore, their cohomology on X_i coincide with cohomology on X_{i-1}. So, from the long exact cohomology sequence and induction hypothesis we get for all $n > n_0$ and all $l > 0$ $H^l(X_i, \mathcal{F} \otimes \mathcal{O}_{X_i}(np_{u,i})) = 0$.

□

The theorem is proved.

□

Remark 14. The constructions of subspaces and geometric data given in the theorem are generalizations of the Krichever map constructed in the works [15], [12]. If a geometric data is taken on a ribbon that comes from a surface and a reduced effective Cartier divisor on it, as in example 1, then these constructions coincide.

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